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DYNAMIC BEHAVIOUR of RIGID CONCRETE PAVEMENTS UNDER DYNAMIC TRAFFIC LOADS

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ABSTRACT : This paper examines the dynamic behaviour of rigid concrete pavements under dynamic traffic loads, which includes the determination of the forces in the concrete plate and in the steel connecting devices at the joints, consisting of dowels and tie bars. For this purpose the rectangular plate is modelled as an elastic homogeneous orthotropic plate supported by a continuous Pasternak foundation, with boundary supports provided by the steel dowels and tie bars, providing elastic vertical support and rotational restraint. The free vibration problem is solved using two transcedental equations, obtained from the solution of two auxiliary Levy's type problems, known as the Modified Bolotin Method. The transcedental equations have infinite number of roots, of which the real roots are the wave numbers, while the integer part of the wave numbers represents the mode numbers. The mode shape is represented as a product of eigenfunctions, which are further used in the dynamic response analysis. The dynamic moving traffic load is expressed as a concentrated load of harmonically varying magnitude, moving on the plate in an arbitrary direction with a constant velocity. The homogeneous solution of the problem is obtained by a method of seperation of variables, in such a way that superposition yields a solution satisfying the boundary conditions. The general solution of the response of the plate to the dynamic moving load in integral form is obtained from the specific properties of the Dirac-delta function, so that it can be further integrated to obtain the various plate response equations during the time interval the load is moving within the plate boundaries, as well as after the load has left the plate. All of the equations are then used to analyse deflections and forces in the concrete plate, including forces in the load transferring steel devices at the joints between consecutive plates. A numerical example is given illustrating the dynamic behaviour of a rigid concrete pavement under a dynamic traffic load.

KEYWORDS: Rigid pavement, dynamic traffic load, dynamic response.

1. INTRODUCTION

The dynamic response of rigid concrete pavements to dynamic moving traffic loads has been studied quite extensively by Alisjahbana, Wangsadinata and Baadilla in recent years [1] [2] [3] [4] [5] [6] [7] [8].

The rigid concrete pavement has been modelled as a rectangular damped orthotropic plate resting on a continuous elastic foundation with side supports providing certain restraint conditions. The dynamic traffic load has been modelled as an equivalent concentrated load of harmonically varying magnitude, moving on the plate with a constant velocity.

For the continuous elastic foundation, several types have been considered, such as proposed by Winkler, Pasternak and Kerr. For the side support restraint conditions, beside vertical support, also rotational restraint conditions have been considered, from simple support, partially fixed to fully fixed (clamped) conditions. In a recent study, Baadilla has formulated the side support restraints, provided at the joints by the steel connecting devices, as actually applied in real concrete pavement construction. These devices consist of steel dowels and steel tie bars, providing elastic vertical support and elastic rotational restraint, depending on the applied number and size of the dowels and tie bars.

In the application of the theory of dynamic response of the orthotropic plate, the continuous elastic foundation modelled as a Pasternak foundation is representing closely the actual subsoil condition, but requiring advanced analytical treatment in solving the dynamic response problem. A Pasternak foundation model incorporates shear interaction between spring elements, mobilized through a plate placed on top of the springs, which deforms only by transverse shear. Thus, in this model compressive and shear deformation of the soil are duly simulated. Using this Pasternak foundation model, for the case of a free vibrating simply supported rectangular plate, the number of waves is represented by mπ/a and nπ/b, where 'a' and 'b' are the length of the plate in the two perpendicular directions and 'm' and 'n' are positive integers, defining the mode number. For the case of the rectangular plate with arbitrarily side support conditions, the number of waves is represented by pπ/a and qπ/b, where 'p' and 'q' are real numbers to be solved from a system of two transcedental equations, obtained from two auxiliary Levy's type problems, also known as the Modified Bolotin Method. The homogeneous solution of the problem is obtained by a method of separation of variables, in such a way that superposition yields a solution satisfying the boundary conditions. As the mode shapes are expressed as products of eigenfunctions, the solution of the dynamic problem is obtained on the basis of orthogonality properties of eigenfunctions. The general solution of the response of the plate to the dynamic moving load in integral form is obtained from the specific properties of the Dirac-delta function, so that it can be further integrated to obtain the various plate response equations during the time internal the load is moving within the plate boundaries, as well as after the load has left the plate. This paper will give an overview of the dynamic response analysis of rigid concrete pavements as described above.

2. SIDE SUPPORT RESTRAINT

Side support restraints are provided by the dowels along the transverse joints and by the tie bars along the longitudinal joints. Dowels are intended to allow longitudinal movements of the concrete plate to occur, while tie bars are intended to prevent cracking of concrete caused by temperature changes. Although each steel connecting device has different functions, both provide similar side support restraints to the concrete plate, which are elastic vertical supports and elastic rotational restraints.

Based on a study by Friberg referred to by Baadilla [8], the elastic vertical support stiffness per unit length of the plate's side f is determined by the following expression:

$$f = \frac{1}{S_d\left(\frac{g}{G_d\, A_d} + \frac{2+\beta_d\, g}{2\,\beta_d^3\, E_d\, I_d}\right)} \qquad \text{.................................... (1)}$$

where S_d is the steel bar spacing, g is the width of the gap between the two adjoining concrete plates, G_d is the steel shear modulus, A_d is the bar cross section area, E_d is the steel modulus of elasticity, I_d is the steel bar cross section moment of inertia, and β_d is a rigidity factor depending on the steel bar diameter and the embedment length of the bar in the concrete plate. For a 16 mm, 22 mm and 25 mm diameter steel bars, the range of the value of β_d is as follows :

$$20 < \beta_d < 40 \qquad \text{.................................... (2)}$$

The elastic rotational stiffness per unit length of the plate's side c is relatively very small, as it is partly mobilized by the steel bars, located at mid depth of the concrete plate and partly by the interlocking effect of the adjoining concrete plates, limiting the related side restraining moment due to traffic loads acting at the steel bars to be far below their yield moment capacity $M_y = 1/32\ (\pi\ d_d^3)\ f_y$, where d_d is the steel bar diameter and f_y the yield strength of the steel.

The complete model of the rigid concrete pavement under dynamic traffic loading is as shown on Figure 1.

3. THE GOVERNING EQUATIONS

Based on the classical theory of thin plates, the governing forced vibration differential equation of a thin damped orthotropic plate resting on a continuous elastic Pasternak foundation is as follows:

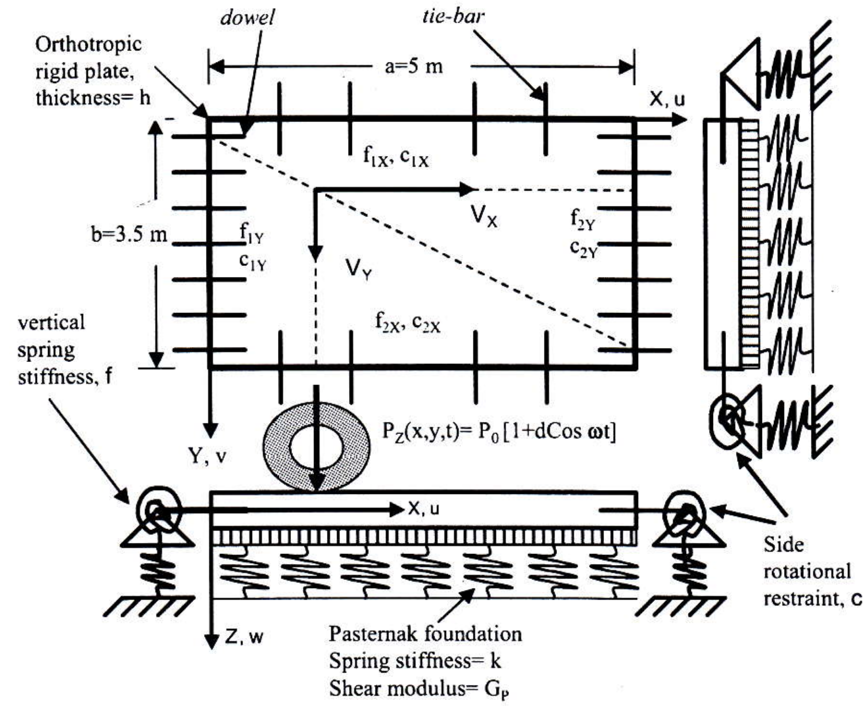

Figure 1. Model of the rigid concrete plate under dynamic traffic loading.

$$D_x\left(\frac{\partial^4 w}{\partial x^4}\right)+2B\frac{\partial^4 w}{\partial x^2\partial y^2}+D_y\left(\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2 w}{\partial t^2}+\gamma h\frac{\partial w}{\partial t}+kw-G_P\nabla^2 w=p_Z(x,y,t)\quad\text{.............}\ (3)$$

where $w(x,y,t)$ = transverse deflection; ρ= plate mass density per unit volume; h= plate thickness; t is the time; γ is the damping ratio; k= spring stiffness and G_P= shear modulus of the Pasternak foundation; $p_Z(x,y,t)$= dynamic load on the plate; D_x, D_y= plate flexural rigidities in the x and y direction, B= effective torsional rigidity, all of which according to the following expressions:

$$D_x=\frac{E_x h^3}{12(1-\nu_x\nu_y)};\quad D_y=\frac{E_y h^3}{12(1-\nu_x\nu_y)};\quad B=D_x\nu_y+\frac{Gh^3}{6}\quad\text{..................................}\ (4)$$

in which E_x and E_y are elasticity moduli along the x and y axes; G is the rigidity modulus; υ_x and υ_y are Poisson's ratios along the x and y axes.

The dynamic load p_Z (x,y,t) modelled as an equivalent concentrated load of harmonically varying magnitude moving in the x and y directional axes of the plate as shown in Figure 1 can be expressed as follows:

$$P_Z(x,y,t) = p\,[x(t),y(t),t] = P(t)\,\delta\,[x-x(t)]\,\delta[y-y(t)]\quad\text{..................................}\ (5)$$

$$P(t)= P_0+\overline{P} = P_0(1+dCos\omega t)\quad\text{..................................}\ (6)$$

$$x(t)= v_1 t \text{ and } y(t)= v_2 t$$

.... (7)

where $\delta[.]$= Dirac-delta function; $x(t)= (v_1 t)$= position function of the load with speed v_1 in the x direction; $y(t)= (v_2 t)$= position function of the load with speed v_2 in the y direction; $P(t)$= concentrated load of harmonically varying magnitude; P_0= mean amplitude of load; $\overline{P}$ = additional varying load due to the interaction effect of the vehicle suspension system, road roughness and speed of the vehicle; d = dynamic load coefficient, which in this paper is taken equal to 0.3; ω= angular frequency of the load.

4. BOUNDARY CONDITIONS

Due to the use of dowels and tie bars to join the concrete pavement plates, all four sides of the plate have elastic vertical translational support as well as elastic rotational restraint along the sides. Thus, the boundary conditions for each side of the plate are as follows:

Elastic vertical support along x=0:

$$V_{(x=0)} = D_x\left[\frac{\partial^3 w(x,y,t)}{\partial x^3}\right]\left(\frac{B+2G_{xy}}{D_x}\right)\frac{\partial^3 w(x,y,t)}{\partial x\,\partial y^2} = f_{1y}\, w(x,y,t)$$

Elastic vertical support along x=a:

$$V_{(x=a)} = D_x\left[\frac{\partial^3 w(x,y,t)}{\partial x^3}\right]\left(\frac{B+2G_{xy}}{D_x}\right)\frac{\partial^3 w(x,y,t)}{\partial x\,\partial y^2} = f_{2y}\, w(x,y,t)$$

Elastic vertical support along y=0:

$$V_{(y=0)} = D_y\left[\frac{\partial^3 w(x,y,t)}{\partial x^3}\right]\left(\frac{B+2G_{xy}}{D_y}\right)\frac{\partial^3 w(x,y,t)}{\partial x\,\partial y^2} = f_{1x}\, w(x,y,t)$$

Elastic vertical support along y=b:

$$V_{(y=b)} = D_y\left[\frac{\partial^3 w(x,y,t)}{\partial y^3}\right]\left(\frac{B+2G_{xy}}{D_y}\right)\frac{\partial^3 w(x,y,t)}{\partial y\,\partial x^2} = f_{2x}\, w(x,y,t) \quad ....(8)$$

where f_{1y}, f_{2y}, f_{1x} and f_{2x} are the elastic vertical translational stiffnesses of the support along the sides of the plate, in [N/m/m], and

Elastic rotational restraint along x=0: $M_{(x=0)} = -D_x\left[\frac{\partial^2 w}{\partial x^2} + v_y\frac{\partial^2 w}{\partial y^2}\right] = c_{1y}\frac{\partial w}{\partial x}$

Elastic rotational restraint along x=a: $M_{(x=a)} = -D_x\left[\frac{\partial^2 w}{\partial x^2} + v_y\frac{\partial^2 w}{\partial y^2}\right] = c_{2y}\frac{\partial w}{\partial x}$

Elastic rotational restraint along y=0: $M_{(y=0)} = -D_y\left[\frac{\partial^2 w}{\partial y^2} + v_x\frac{\partial^2 w}{\partial x^2}\right] = c_{1x}\frac{\partial w}{\partial y}$

Elastic rotational restraint along y=b: $M_{(y=b)} = -D_y\left[\frac{\partial^2 w}{\partial y^2} + v_x \frac{\partial^2 w}{\partial x^2}\right] = c_{2x}\frac{\partial w}{\partial y}$ .... (9)

where c_{1y}, c_{2y}, c_{1x} and c_{2x} are the elastic rotational stiffnesses of the support along the sides of the plate, in [Nm/rad/m]. It is assumed that the principal elastic axes x,y of the material are parallel to the plate's sides.

5. GENERAL ANALYSIS

The dynamic deflection response w(x,y,t) of a plate with arbitrary support conditions is the general solution of Eq.(3), which consists of the homogeneous free vibration solution w_H and the particular forced vibration solution w_P thus:

$$w(x,y,t) = w_H + w_P \qquad (10)$$

The homogeneous free vibration solution of the thin elastic orthotropic plate resting on a continuous Pasternak foundation is governed by the homogeneous form of Eq.(3). To obtain the homogeneous solution w_H, the Modified Bolotin Method is used, with which firstly we have to find the free vibration natural frequencies of a simply supported plate. For that purpose it is assumed that the free vibration solution of a simply supported plate will take the following form :

$$w(x,y,t) = W(x,y)\ \text{Sin}\ \omega t \qquad (11)$$

in which the natural modes shape function W(x,y) is a function of the coordinates (x,y) only. By substituting Eq.(11) into the undamped homogeneous form of Eq.(3) the following equation is obtained:

$$D_x\left(\frac{\partial^4 w(x,y,t)}{\partial x^4}\right) + 2B\frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + D_y\left(\frac{\partial^4 w(x,y,t)}{\partial y^4}\right) - \rho h\frac{\partial^2 w(x,y,t)}{\partial t^2} +$$
$$+ kw(x,y,t) - G_p\left(\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2}\right) = 0 \qquad (12)$$

For the plate with all edges simply supported, the following expression will obviously satisfy the boundary conditions:

$$W_{mn}(x,y,t) = A_{mn}\ \text{Sin}\frac{m\pi x}{a}\frac{n\pi y}{b}\ \text{Sin}\ \omega t \qquad (13)$$

where A_{mn} is an amplitude coefficient determined from the initial conditions, m and n being positive integers and ω the natural circular frequency of vibration. Substituting Eq.(13) into Eq.(12) gives the natural circular frequency of the system expressed as $(\omega_{mn})^2$ as follows:

$$(\omega_{mn})^2 = \frac{1}{\rho h}\left[k + G_P\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} + D_x\left(\frac{m\pi}{a}\right)^4 + 2B\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_y\left(\frac{n\pi}{b}\right)^4\right]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (14)$$

The next step is to find the solution of Eq.(12) and obtain the eigen-frequencies and mode-shapes of the orthotropic plate with arbitrary side support conditions according to Eq.(8) and Eq.(9). By postulating the following natural circular frequency equation, analogous to the case of a simply supported plate at all edges, Eq.(14) can also be expressed as:

$$(\omega_{mn})^2 = \frac{1}{\rho h}\left[k+G_p\left\{\left(\frac{p\pi}{a}\right)^2+\left(\frac{q\pi}{b}\right)^2\right\}+D_x\left(\frac{p\pi}{a}\right)^4+2B\left(\frac{p\pi}{a}\right)^2\left(\frac{q\pi}{b}\right)^2+D_y\left(\frac{q\pi}{b}\right)^4\right] \quad \ldots\ldots\ldots\ldots\ (15)$$

where p and q are real numbers, such that $m\leq p \leq m+1$ and $n\leq q \leq n+1$, to be solved from a system of two transcendental equations, obtained from the solution of the next two auxiliary Levy's type problems. This procedure is known as the Modified Bolotin Method.

6. FIRST AUXILIARY LEVY'S TYPE PROBLEM IN THE x DIRECTION

The solution of the first auxiliary problem of Eq.(12) satisfying the boundary conditions according to Eq.(8) and Eq.(9) can be assumed to have the following form:

$$w(x,y) = X(x)\ \mathrm{Sin}\frac{q\pi y}{b} \quad \ldots\ldots\ldots\ldots\ (16)$$

and substituting Eq.(16) into the undamped partial differential equation according to Eq. (12) yields a fourth order differential equation in X(x) as follows:

$$\frac{\partial^4 X}{\partial X^4}-\left\{\frac{2B}{D_x}\left(\frac{q\pi}{b}\right)^2+\frac{G_p}{D_x}\right\}\frac{\partial^2 X}{\partial x^2}+\left\{\frac{D_y}{D_x}\left(\frac{q\pi}{b}\right)^4+\frac{G_p}{D_x}\left(\frac{q\pi}{b}\right)^2+\frac{k}{D_x}-\frac{\rho h}{D_x}\omega^2\right\}X=0 \quad \ldots\ldots\ldots\ldots\ (17)$$

for which the corresponding characteristic equation is obtained by substituting $X(x)= Ae^{\lambda x}$ into Eq.(17), which is as follows:

$$\lambda^4-\left\{\frac{2B}{D_x}\left(\frac{q\pi}{b}\right)^2+\frac{G_P}{D_x}\right\}\lambda^2-\left(\frac{p\pi}{a}\right)^2\left[\frac{2B}{D_x}\left(\frac{q\pi}{b}\right)^2+\left(\frac{p\pi}{a}\right)^2+\frac{G_P}{D_x}\right]= 0 \quad \ldots\ldots\ldots\ (18)$$

This characteristic equation possesses two imaginary and two real roots; namely:

$$\lambda_{1,2}=\pm\left(\frac{p\pi}{a}\right)i=\pm(k_1)i \text{ and } \lambda_{3,4}=\pm\sqrt{\frac{G_P}{D_x}+\left(\frac{p\pi}{a}\right)^2+\frac{2B}{D_x}\left(\frac{q\pi}{b}\right)^2}=\pm k_3 \quad \ldots\ldots\ldots\ (19)$$

Thus, the solution of the first auxiliary Levy's type problem in the x direction, which is the general solution of Eq.(17) can be represented as:

$$X_{pq}(x) = A_1\mathrm{Cos}(k_1 x)+ A_2\mathrm{Sin}(k_1 x) + A_3\mathrm{Cosh}(k_3 x) + A_4\mathrm{Sinh}(k_3 x) \quad \ldots\ldots\ldots\ (20)$$

Substitution of Eq.(20) into the boundary conditions for the sides of the plate at x=0 and x=a according to Eq.(8) and Eq.(9) will produce 4 boundary equations, which can be put into a matrix equation as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{.................................. (21)}$$

The requirement for the existence of a non-trivial solution yields the first transcendental equation:

$$\mathrm{Det} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = 0 \qquad \text{.................................... (22)}$$

and the linearly dependent coefficients A_i of the mode shape function X(x) according to Eq.(20) as follows:

$$A_4 = -\frac{a_{11}}{a_{14}}A_1 - \frac{a_{12}}{a_{14}}A_2 - \frac{a_{13}}{a_{14}}A_3 \alpha = A_{11} + \alpha_1 A_{12} + \alpha_2 A_{13}$$

$$A_3 = -\frac{a_{21} + \alpha_{11}a_{24}}{a_{23} + \alpha_{13}a_{24}}A_1 - \frac{a_{22} + \alpha_{12}a_{24}}{a_{23} + \alpha_{13}a_{24}}A_2 \alpha = A_{21} + \alpha_1 A_{22}$$

$$A_2 = -\frac{a_{31} + \alpha_{11}a_{34} + \alpha_{21}(a_{33} + \alpha_{13}a_{34})}{a_{32} + \alpha_{12}a_{34} + \alpha_{22}(a_{33} + \alpha_{13}a_{34})}A_1$$

or

$$A_2 = -\frac{a_{41} + \alpha_{11}a_{44} + \alpha_{21}(a_{43} + \alpha_{13}a_{44})}{a_{42} + \alpha_{11}a_{44} + \alpha_{22}(a_{43} + \alpha_{13}a_{44})}A_1 \qquad \text{............................... (23)}$$

7. SECOND AUXILIARY LEVY'S TYPE PROBLEM IN THE y DIRECTION

The solution of the second auxiliary problem of Eq.(12) satisfying the boundary conditions according to Eq.(8) and Eq.(9) can be assumed to have the following form:

$$w(x,y) = Y(y)\ \mathrm{Sin}\frac{p\pi x}{a} \qquad \text{............................... (24)}$$

By following the same procedure as in the derivation of Eq.(18), the following characteristic equation is obtained:

$$\lambda^4 - \left\{ \frac{2B}{D_y}\left(\frac{p\pi}{a}\right)^2 + \frac{G_P}{D_y} \right\}\lambda^2 - \left(\frac{q\pi}{b}\right)^2 \left[\frac{2B}{D_y}\left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2 + \frac{G_P}{D_y} \right] = 0 \quad \text{(25)}$$

which also possesses two imaginary and two real roots, namely :

$$\lambda_{1,2} = \pm\left(\frac{q\pi}{b}\right)i = \pm (k_2)i \text{ and } \lambda_{3,4} = \pm\sqrt{\frac{G_P}{D_y} + \left(\frac{q\pi}{b}\right)^2 + \frac{2B}{D_y}\left(\frac{p\pi}{a}\right)^2} = \pm k_4 \quad \text{(26)}$$

which will yield the solution of the second auxiliary Levy's type problem in the y direction:

$$Y_{pq}(y) = B_1 Cos(k_2 y) + B_2 Sin(k_2 y) + B_3 Cosh(k_4 y) + B_4 Sinh(k_4 y) \quad \text{(27)}$$

Substitution of Eq.(27) into the boundary conditions for the sides of the plate at y=0 and y=b according to Eq.(8) and Eq.(9), will produce 4 boundary equations, which can be put into a matrix equation as follows:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{(28)}$$

The requirement for the existence of a non-trivial solution yields the second transcendental equation:

$$\text{Det}\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = 0 \quad \text{(29)}$$

and the linearly dependent coefficients B_i of the mode shape function Y(y) according to Eq.(27) as follows:

$$B_3 = -\frac{b_{21} + \beta_{11} b_{24}}{b_{23} + \beta_{13} b_{24}} B_1 - \frac{b_{22} + \beta_{12} b_{24}}{b_{23} + \beta_{13} b_{24}} B_2 \beta = B_{21} + \beta_1 B_{22}$$

$$B_2 = -\frac{b_{31} + \beta_{11} b_{34} + \beta_{21}(b_{33} + \beta_{13} b_{34})}{b_{32} + \beta_{12} b_{34} + \beta_{22}(b_{33} + \beta_{13} b_{34})} B_1$$

or

$$B_2 = -\frac{b_{41} + \beta_{11} b_{44} + \beta_{21}(b_{43} + \beta_{13} b_{44})}{b_{42} + \beta_{11} b_{44} + \beta_{22}(b_{43} + \beta_{13} b_{44})} B_1$$

8. NATURAL FREQUENCIES AND NATURAL MODES

The characteristic determinants expressed by Eq.(22) and Eq.(29) being transcendental in nature, have an infinite number of roots p and q. This unknown quantities p and q are calculated from the solution of these equations. By substituting p and q into Eq.(15), the natural circular frequencies of the plate with arbitrary support conditions can be obtained. The integer part of the real roots p and q represents the number of the natural circular frequency. The natural modes are determined as the following product:

$$w_{pq} = X_{pq}(x)Y_{pq}(y) \qquad \text{.............................. (31)}$$

where $X_{pq}(x)$and $Y_{pq}(y)$ are according to Eq.(20) and Eq.(27) respectively.

9. HOMOGENEOUS SOLUTION w_H

The general homogeneous solution of the partial differential Eq.(3) can be obtained by a method of separation of variables. This technique is particularly useful for the direct solution of boundary value problems, where the boundary conditions have a simple form. The procedure comprises the derivation of a sequence of solutions of a separable form, in such a way that superposition yields a solution satisfying the boundary conditions. According to this method, the general homogeneous solution of Eq.(3) is set to be separated into a time domain function and spatial functions as follows:

$$w_H = w_{pq}(x,y,t) = w(x,y)T(t) = X_{pq}(x)\; Y_{pq}(y)\; T(t) = X\;Y\;T \qquad \text{............................ (32)}$$

Substitution of Eq.(32) into the homogeneous form of the partial differential Eq.(3) will separate the differential equation by a separational constant β, into two partial differential equations as follows:

$$\ddot{T} + \frac{\gamma h}{\rho h}\dot{T} + \frac{\beta}{\rho h}T = 0 \qquad \text{.............................. (33)}$$

$$D_x\frac{X^{IV}}{X} + 2B\frac{X^{II}Y^{II}}{XY} + D_y\frac{Y^{IV}}{Y} + k - G_p\left(\frac{X^{II}}{X} + \beta\frac{Y^{II}}{Y}\right) = 0 \qquad \text{.............................. (34)}$$

The separational constant β can be obtained by substituting $X=A_0\text{Sin}[p\pi x/a]$ and $Y=B_0\text{Sin}[q\pi y/b]$ into Eq.(34) and after rearranging for β, the result is as follows:

$$\beta=\left[k+G_P\left\{\left(\frac{p\pi}{a}\right)^2+\left(\frac{q\pi}{b}\right)^2\right\}+D_x\left(\frac{p\pi}{a}\right)^4+2B\left(\frac{p\pi}{a}\right)^2\left(\frac{q\pi}{b}\right)^2+D_y\left(\frac{q\pi}{b}\right)^4\right] \text{.................(35)}$$

The linear second order differential equation Eq.(33) has the form of the already well known damped free vibration system. The frequencies of the damped free vibration system can be obtained and can be defined as follows:

$$\omega_{pq}^{\;2} = \frac{\beta}{\rho h} \qquad \text{........................ (36)}$$

By assuming that viscous damping only is present in the system, the following relationship applies:

$$\zeta = \frac{c}{c_r} = \frac{c}{2m\omega_{pq}} \qquad \text{......................... (37)}$$

where ζ = damping ratio; c = damping constant, which is a measure of the energy dissipated in a complete cycle of vibration; c_r = critical damping.

By using Eq.(36) and Eq.(37), the linear homogeneous form of the second order differential equation

Eq.(33) can be put in the following form:

$$\ddot{T} + 2\zeta\omega_{pq}\,\dot{T} + \omega_{pq}^{\;2}\,T = 0 \qquad (38)$$

The substitution of a solution in the form of $T(t)=e^{St}$ into Eq.(38), with the constant S still unknown, yield the roots $S_{1\,\&\,2}$ of the characteristic equation as follows:

$$S^2 + 2\zeta\omega_{pq}S + \left(\omega_{pq}\right)^2 = 0 \text{ and the roots } S_{1\&2} = \omega_{pq}\left(-\zeta \pm i\sqrt{1-\zeta^2}\right) \qquad (39)$$

so that the temporal function T(t) can be obtained as follows:

$$T_{pq}(t)= A_0\, e^{S_1 t} + B_0\, e^{S_2 t} = e^{-\zeta\omega_{pq}t}\left(a_0 e^{i\sqrt{1-\zeta^2}\omega_{pq}t} + b_0 e^{-i\sqrt{1-\zeta^2}\omega_{pq}t}\right) =$$
$$= e^{-\zeta\omega_{pq}t}\left(a_0 \mathrm{Cos}\omega_D t + b_0 \mathrm{Sin}\omega_D t\right) \qquad (40)$$

where the constants a_0 and b_0 can be determined from the initial conditions.

Since the spatial functions $X_{pq}(x)$ and $Y_{pq}(y)$, and the temporal function $T_{pq}(t)$ are already obtained, the homogeneous solution of the forced vibration differential equation is the product of the two functions as follows:

$$W_{pq}(x,y,t)=X_{pq}Y_{pq}T(t) = [X_{pq}(x)Y_{pq}(y)]\, e^{-\zeta\omega_{pq}t}\left(a_{0pq} e^{i\sqrt{1-\zeta^2}\omega_{pq}t} + b_{0pq} e^{-i\sqrt{1-\zeta^2}\omega_{pq}t}\right) \qquad (41)$$

Using the orthogonality property of the eigen functions the total elastic response of the system is obtained by superposing deformations of the modes in the form of series expansions. The complete homogeneous solution of the partial differential equation Eq.(3) for each m=p and n=q can finaly be expressed as:

$$w_H = \sum_{m=[p]=1}^{\infty}\sum_{n=[q]=1}^{\infty}\left[X_{pq}(x)Y_{pq}(y)\right] e^{-\zeta\omega_{pq}t}\left(a_{0pq} e^{i\sqrt{1-\zeta^2}\omega_{pq}t} + b_{0pq} e^{-i\sqrt{1-\zeta^2}\omega_{pq}t}\right) =$$
$$= \sum_{m=[p]=1}^{\infty}\sum_{n=[q]=1}^{\infty}\left[X_{pq}(x)Y_{pq}(y)\right] e^{-\zeta\omega_{pq}t}\left(a_0 \mathrm{Cos}\omega_D t + b_0 \mathrm{Sin}\omega_D t\right) \qquad (42)$$

where $\omega_D = \sqrt{1-\zeta^2}\,\omega_{pq}$

10. DYNAMIC RESPONSE OF THE PLATE

Since a fundamental set of solutions of the homogeneous partial diffrentail equation is known and given by the eigenfunctions, it is appropriate to use the method of variation of parameters as a general method of determining a particular solution w_P of the corresponding non-homogeneous partial differential equation, which can be written in the following form:

$$w_P = w_{pq}(x,y,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} X_{pq}(x)Y_{pq}(y)T_{pq}(t) \qquad (43)$$

where $T_{pq}(t)$ is the temporal function, which must be determined through further analysis.

The differential equation for the temporal function $T_{pq}(t)$ can be expressed as:

$$\ddot{T}_{pq}(t) + 2\zeta\omega_{pq}\,\dot{T}_{pq}(t) + \omega_{pq}{}^{2}\cdot T_{pq}(t) = \int_{x=0}^{a}\int_{y=0}^{b} X_{pq}(x)Y_{pq}(y)\frac{P_z(x,y,t)}{\rho hQ_{pq}}dxdy \quad \text{.............. (44)}$$

where $P_Z(x,y,t)$ is an arbitrary dynamic load and Q_{pq} a normalization factor.

Note that the homogeneous solution of Eq.(44) is identical with the solution of the homogeneous differential equation Eq.(38). The total solution of Eq.(44) consists of the homogeneous solution and a particular solution:

$$T_{pq}(t) = \hat{T}_{pq}(t) + T^{*}_{pq}(t) \quad \text{......................... (45)}$$

where $\hat{T}_{pq}(t)$ is the homogeneous solution as expressed by Eq.(40) and $T^{*}_{pq}(t)$ the particular solution which can be represented in a form of the Duhamel's convolution integral as follows:

$$T^{*}_{pq}(t) = \int_{0}^{t}\left[\frac{P_z(x,y,t)}{\rho hQ_{pq}}\int_{x=0}^{a}X_{pq}(x)dx\int_{y=0}^{b}Y_{pq}(y)dy\frac{e^{-\zeta\omega_{pq}(t-\tau)}}{\sqrt{1-\zeta^{2}}\,\omega_{pq}}Sin\sqrt{1-\zeta^{2}}\,\omega_{pq}(t-\tau)\right]d\tau \quad \text{....... (46)}$$

The homogeneous solution $\hat{T}_{pq}(t)$ contains constants that must be determined. The nature of the steady state forced responses of the plate is contained entirely in the functions $T^{*}_{pq}(t)$ defined by Eq.(46). Substituting the expressions for the coefficient functions in Eq.(46), the general solution for the forced response deflection of the plate to an arbitrary dynamic load $P_Z(x,y,t)$ is given in integral form as follows:

$$w_{pq}(x,y,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}X_{pq}(x)Y_{pq}(y)\Bigg[e^{-\zeta\omega_{pq}t}\left[a_{0pq}Cos[\sqrt{1-\zeta^{2}}\,\omega_{pq}t] + b_{0pq}Cos[\sqrt{1-\zeta^{2}}\,\omega_{pq}t]\right] + e^{-\zeta\omega_{pq}t}$$

$$\left[\int_{0}^{t}\frac{P_z(x,y,t)}{\rho hQ_{pq}}\int_{x=0}^{a}X_{pq}(x)dx\int_{y=0}^{b}Y_{pq}(y)dy\frac{e^{\zeta\omega_{pq}\tau}}{\sqrt{1-\zeta^{2}}\,\omega_{pq}}Sin\sqrt{1-\zeta^{2}}\,\omega_{pq}(t-\tau)d\tau\right]\Bigg] \quad \text{................... (47)}$$

The general equation presented above may be integrated to determine the response of the plate to an arbitrary dynamic load $P_Z(x,y,t)$.

In Eq.(47) the constants a_{0pq} and b_{0pq} must be determined from the initial conditions at $t=t_0$ and represent a transient state vibratory motion resulting from the initial conditions. In the remainder of this paper, attention will be centered on the dynamic response of the system subjected to the dynamic load described by the loading functions according to Eq.(5), Eq.(6) and Eq.(7).

Bending moments and vertical shear forces in the plate can be computed in terms of the deflection and its derivatives obtained from Eq.(47) as expressed by the following equations:

Bending moments: $M_x = -Dx\left(\frac{\partial^2 w}{\partial x^2} + \upsilon_y\frac{\partial^2 w}{\partial y^2}\right)$; $M_y = -Dy\left(\frac{\partial^2 w}{\partial y^2} + \upsilon_x\frac{\partial^2 w}{\partial x^2}\right)$

Shear forces: $Q_x = -\frac{\partial}{\partial x}\left(D_x\frac{\partial^2 w}{\partial x^2} + B\frac{\partial^2 w}{\partial y^2}\right)$; $Q_y = -\frac{\partial}{\partial y}\left(D_y\frac{\partial^2 w}{\partial y^2} + B\frac{\partial^2 w}{\partial x^2}\right)$

Vertical edge forces: $V_x = -D_x\left[\frac{\partial^3 w}{\partial x^3} + H_x\frac{\partial^3 w}{\partial x\partial y^2}\right]$; $V_y = -D_y\left[\frac{\partial^3 w}{\partial y^3} + H_y\frac{\partial^3 w}{\partial x^2\partial y}\right]$

.... (48)

where:

$B = \upsilon_x D_y + 2G_{xy} = \upsilon_y D_x + 2G_{xy}$ is the effective torsional rigidity of the orthotropic plate.

$H_x = \upsilon\frac{4G_{xy}}{D_x} + y = \frac{B+2G_{xy}}{D_x}$ is the plate stiffness constant (vertical side forces) along the sides x=0 and x=a.

$H_y = \upsilon \frac{4G_{xy}}{D_y} + x = \frac{B+2G_{xy}}{D_y}$ is the plate stiffness constant (vertical side forces) along the sides y=0 and y=b.

11. NUMERICAL EXAMPLE

To illustrate the use of the procedure described in this paper, all dowels and tie bars are assumed to have the same elastic vertical translational support and the same elastic rotational restraint. The plate and all other parameters are as follows.

Concrete pavement plate dimension and material:

Length of the plate parallel to the traffic direction a= 5.0 m; width of lane b= 3.5 m; thickness h= 0.25 m; E_x= 27 GPa; E_x= 22. 5 GPa; υ_x= 0.18; υ_y= 0.15; ρ= 2500 kg/m^3.

Pasternak foundation:

Foundation's spring stiffness k= 27.2 MN/m3; foundation's shear modulus G_P= 9.52 MN/m.

Side support restraint of the pavement plate:

Elastic vertical translational stiffness of support along all sides of the plate:

f_{1x}= f_{2x}= 40 MN/m/m; f_{1y}= f_{2y}= 100 MN/m/m. Elastic rotational stiffness of support along the sides of the plate: c_{1x}= c_{2x}= c_{1y}= c_{2y}= 1.0 Nm/rad/m.

Dynamic traffic load:

Wheel equivalent single axle load P_0= 80 kN; dynamic load coefficient d=0.3; travelling velocity in the traffic direction v_l= 72 km/hour = 20 m/sec.; travelling velocity perpendicular to the traffic direction v_2= 2 m/sec.

Using the above data, various wave numbers, natural circular frequencies, deflection response spectra, deflection shapes, deflection time histories, plate and joint internal forces have been computed and some of the results are shown here.

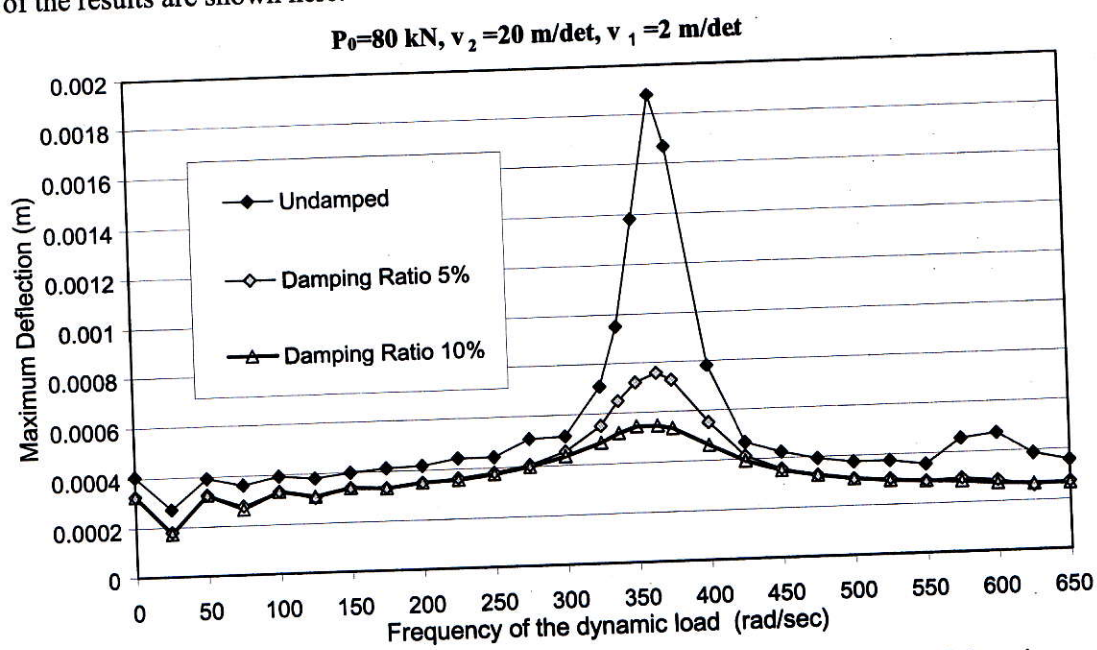

Figure 2. Maximum dynamic deflection response spectra for various values of damping ratio at the center of the plate.

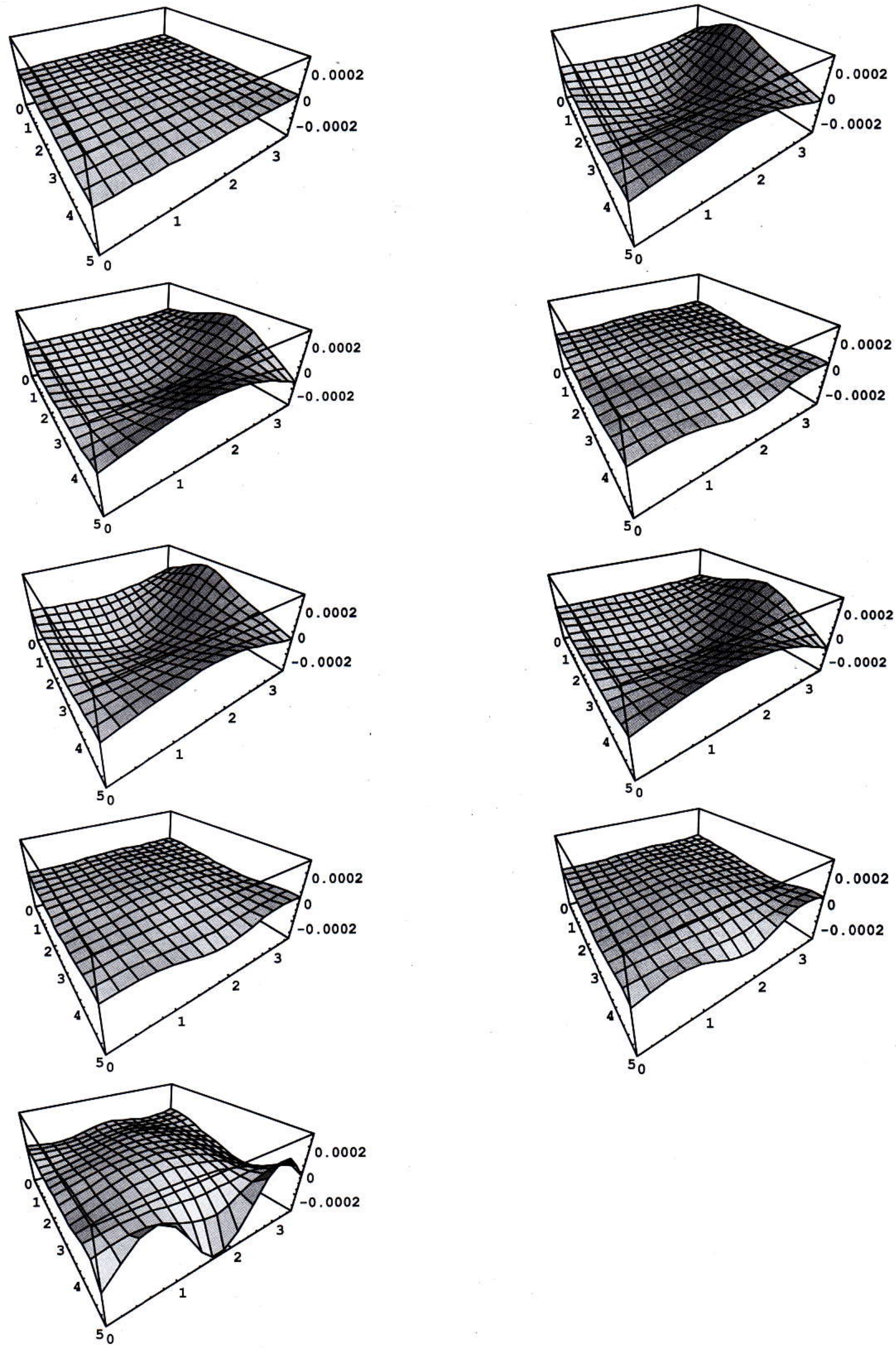

Figure 3. Dynamic deflection surface of the plate due to the moving dynamic traffic load during the interval $0 < t < to = 0{,}25$ sec.; $\zeta = 5\%$, $\omega = 200$ rad/sec.

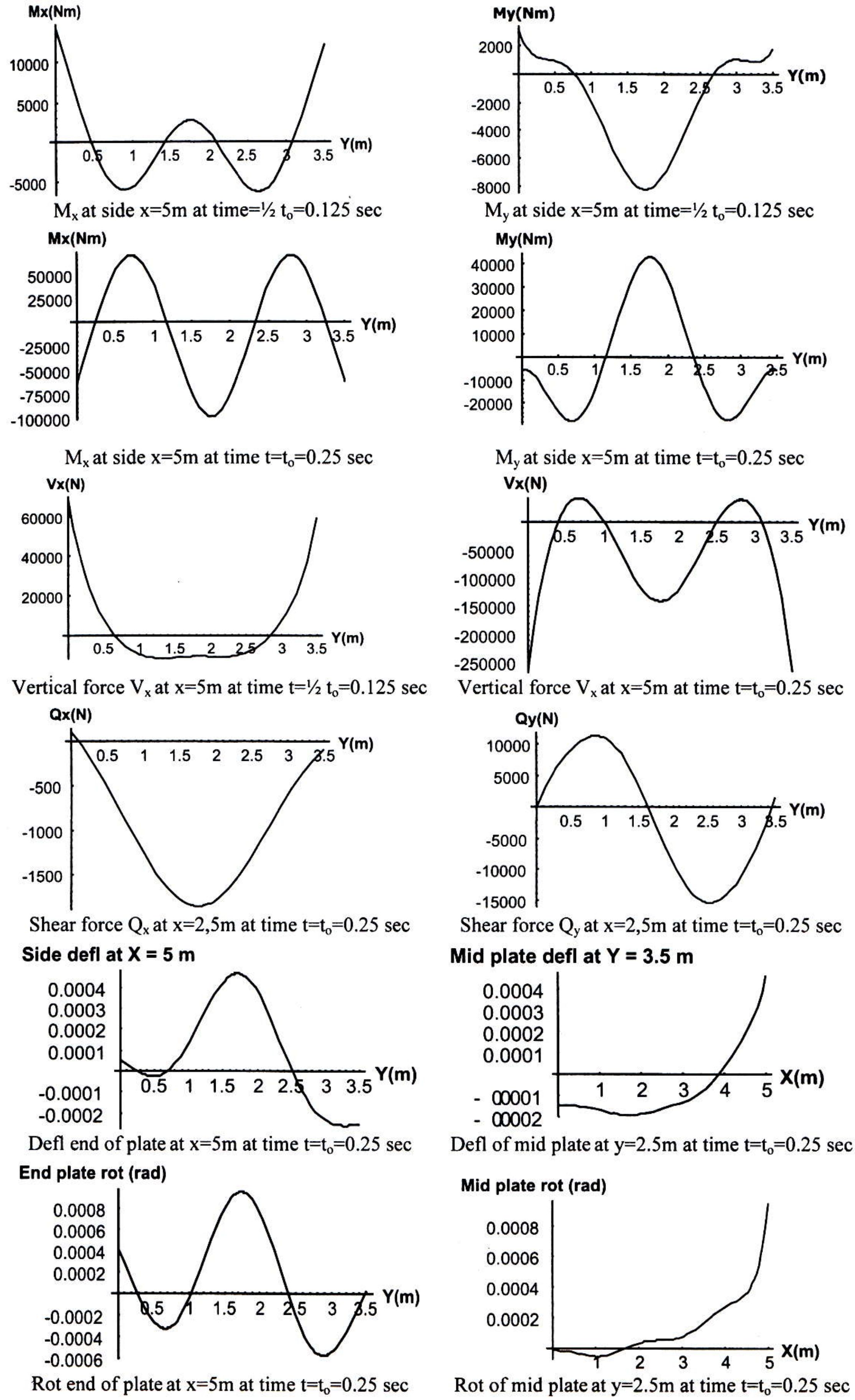

Fig. 4. Various dynamic responses of the plate to the dynamic traffic load; ζ = 5%, ω = 200 rad/sec.

12. CONCLUSION

Based on the study and the numerical example given above it can be concluded, that it is possible to find a mathematical closed solution for the dynamic response of a rectangular rigid concrete pavement plate with arbitrary side support elastic vertical translational and rotational restraints to dynamic traffic loads. The result however should be verified further with the results of experimental research, especially on the determination of the forces in the steel connecting devices at the joints.

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