13th World Conference on Earthquake Engineering
Vancouver, B.C., Canada
August 1-6, 2004
Paper No. 3176

DYNAMIC RESPONSE OF CLAMPED ORTHOTROPIC PLATES TO DYNAMIC MOVING LOADS

Sofia W. ALISJAHBANA[1]

SUMMARY

An approximate method is presented for the determination of the natural frequencies and mode shapes of rectangular clamped orthotropic plates subjected to dynamic moving loads. The natural frequencies of a clamped plate are presented in a form analogous to the corresponding frequencies of a simply supported plate. The wave numbers are the unknown quantities that can be determined from a system of two transcendental equations, obtained from the solution of two auxiliary Levy's type problems. The general solution for the forced response is given in an integral form based on Duhamel's method. A numerical example is further discussed.

INTRODUCTION

Widespread use of filamentary composite materials in several fields of modern technology has made it desirable to investigate the dynamic behavior of plates under the effects of material anisotrophy. Analytical and experimental studies of small deflection free vibration of orthotropic plates had been carried out by many authors. The most comprehensive study had been done by Leissa [5]. An exact solution of the differential equation of a vibrating orthotropic plate had been found for the case of a rectangular plate, simply supported along one pair of opposite edges, known as Levy's problem. The exact solution for the plate with all sides clamped was so far unknown. In the mean time a considerable number of approximate solutions could be found in the literature for several combinations of boundary conditions, including the case of clamped plates. Elishakoff in 1974 [4] investigated the dynamic response of a clamped square orthotropic plate. As a point of departure in his analysis, the frequencies were presented in a form fully analogous to the corresponding frequencies of a simply supported plate. For a simply supported plate, the wave numbers were taken equal to $m\pi/a$ and $n\pi/b$ respectively, where a and b denoted the lengths of the side of the plate and m and n were positive integers, determining the number of mode shapes.

[1] Vice Director School of Graduate Studies, Tarumanagara University, Jakarta- Indonesia. Email: wangsadi@indosat.net.id

The present analysis deals with the dynamic response of a rectangular clamped orthotropic plate subjected to a dynamic transverse moving load. In this analysis, the wave numbers are adopted from the work done by Elishakoff in 1974. These wave numbers are presented in the form $p\pi/a$ and $q\pi/b$, where the pair of real quantities p and q are to be found from the solution of two supplementary eigenvalue problems. The integer parts of p and q represent frequency numbers. The mode shape is presented as a product of eigenfunctions. The dynamic solution of the plate is based on orthogonality conditions of eigenfunctions similar to those used by Alisjahbana, S.W. [2] in analyzing forced responses of simply supported rectangular orthotropic plates subjected to the dynamic transverse moving load. The dynamic response of the plate can be expressed in integral form that can be readily integrated to determine the plate responses for any applied surface loading p(x,y,t).

ANALYTICAL FORMULATION

Free small amplitude vibrations of a thin, elastic orthotropic plate as shown in Figure 1 are governed by the linear partial differential equation

$$D_x \frac{\partial^4 w}{\partial x^4} + 2B \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + \rho h \frac{\partial^2 w}{\partial t^2} + \gamma h \frac{\partial w}{\partial t} = 0 \tag{1}$$

where D_x and D_y are flexural rigidities and are defined by

$$D_x = \frac{E_x h^3}{12(1-\nu_x \nu_y)}, \quad D_y = \frac{E_y h^3}{12(1-\nu_x \nu_y)}, \quad B = D_x \nu_y + \frac{G h^3}{6} \tag{2}$$

in which E_x and E_y are Young's moduli along the x and y axes respectively, G is the rigidity modulus, υ_x and υ_y are Poisson's ratios for the material, for which $E_x\upsilon_y=E_y\upsilon_x$, ρ is the mass density per unit volume of the plate, h is the plate thickness, t is the time, γ is the damping ratio. It is assumed in Equation (1) that principal elastic axes of the material are parallel to the plate edges.

If the free vibration solution of the problem is set as

$$w(x, y, t) = W(x, y) \sin \omega t \tag{3}$$

where ω is the circular frequency and W(x,y) is a function of the position coordinates only, then by substituting Equation (3) into the undamped form of Equation (1) yields

$$D_x \frac{\partial^4 W}{\partial x^4} + 2B \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} - \rho h \omega^2 W = 0 \tag{4}$$

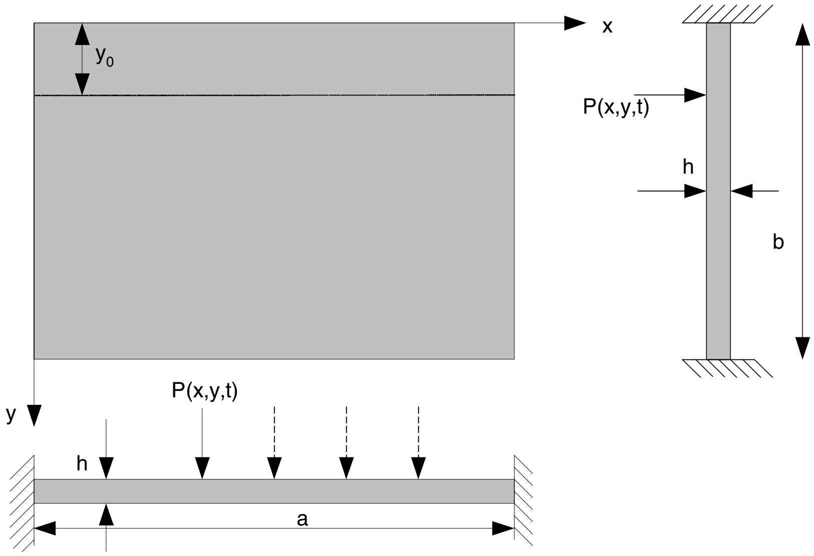


Figure 1. A rectangular clamped orthotropic plate subjected to a dynamic transverse load.

For a plate with all sides simply supported the boundary condition is

$$W(x,y) = \nabla^2 W(x,y) = 0 \tag{5}$$

where ∇^2 is the Laplacian operator.

For a plate with all sides clamped the boundary condition is

$$W(x,y) = \frac{\partial W(x,y)}{\partial \tau} = 0 \tag{6}$$

where τ denotes the direction normal to the contour of the plate.

For the plate with all sides simply supported that satisfies Equation (5), it can be seen that W(x,y) can be expressed as

$$W_{mn}(x,y) = A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{7}$$

where A_{mn} is an amplitude coefficient determined from the initial conditions of the problem and m and n are positive integers. Substituting Equation (7) into Equation (3) gives the natural frequency of the system

$$\omega_{mn} = \left(\frac{B}{\rho h}\right)^{1/2} \left[\frac{D_x}{B}\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \frac{D_y}{B}\left(\frac{n\pi}{b}\right)^4\right]^{1/2} \tag{8}$$

The purpose of this paper is to find the solution of Equation (3) with the boundary conditions according to Equation (6), i.e. the eigen frequencies and the mode shapes of a clamped orthotropic plate. By postulating the following eigen frequency, which is analogous to the case of a plate with all sides simply supported [4], Equation (8) can be expressed as

$$\omega_{mn} = \left(\frac{B}{\rho h}\right)^{1/2}\left[\frac{D_x}{B}\left(\frac{p\pi}{a}\right)^4 + 2\left(\frac{p\pi}{a}\right)^2\left(\frac{q\pi}{b}\right)^2 + \frac{D_y}{B}\left(\frac{q\pi}{b}\right)^4\right]^{1/2} \tag{9}$$

where p and q are real numbers which are to be found.

These two real numbers p and q can be determined by considering two auxiliary problems of Levy's type [4].

First Auxiliary Problem

The solution of Equation (4) for the first auxiliary problem that satisfies the boundary conditions

$$X(x) = \frac{dX(x)}{dx} = 0 \text{ at } x=0,a \tag{10}$$

can be expressed as

$$W(x,y) = X(x)\sin\frac{q\pi y}{b} \tag{11}$$

Substituting Equation (11) into Equation (4) results in an ordinary differential equation for X(x)

$$\frac{d^4X(x)}{dx^4} - 2\frac{q^2\pi^2}{b^2}\frac{B}{D_x}\frac{d^2X(x)}{dx^2} + \left(\frac{q^4\pi^4}{b^4}\frac{D_y}{D_x} - \frac{\rho h\omega^2}{D_x}\right)X(x) = 0 \tag{12}$$

or, given the postulating Equation (9), one obtains

$$\frac{d^4X}{dx^4} - 2\frac{q^2\pi^2}{b^2}\frac{B}{D_x}\frac{d^2X}{dx^2} - \frac{p^2\pi^2}{a^2}\left[2\frac{B}{D_x}\frac{q^2\pi^2}{b^2} + \frac{p^2\pi^2}{a^2}\right]X = 0 \tag{13}$$

The corresponding characteristic equation

$$\lambda^4 - 2\frac{q^2\pi^2}{b^2}\frac{B}{D_x}\lambda^2 - \frac{p^2\pi^2}{a^2}\left[2\frac{B}{D_x}\frac{q^2\pi^2}{b^2} + \frac{p^2\pi^2}{a^2}\right] = 0 \tag{14}$$

which has two imaginary and two real roots, namely

$$\lambda_{1,2} = \pm \frac{\pi}{ab}\kappa_1, \quad \lambda_{3,4} = \pm i\frac{p\pi}{a} \tag{15}$$

where

$$\kappa_1 = \sqrt{\frac{2q^2a^2B}{D_x} + p^2b^2} \tag{16}$$

The general solution of Equation (13) can be represented as

$$X(x) = A_1\cosh(\kappa_1\pi\eta_1) + A_2\sinh(\kappa_1\pi\eta_1) + A_3\cos(p\pi\eta_1 b) + A_4\sin(p\pi\eta_1 b) \tag{17}$$

where A_j are constants of integration and $\eta_1 = \dfrac{x}{ab}$.

When Equation (17) is substituted into Equations (10), the existence of a nontrivial solution yields the characteristic determinant

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & \dfrac{\kappa_1}{b} & 0 & p \\ C_1 & S_1 & c_1 & s_1 \\ \dfrac{\kappa_1}{b}S_1 & \dfrac{\kappa_1}{b}C_1 & -ps_1 & pc_1 \end{vmatrix} = 0 \tag{18}$$

where

$$\begin{aligned} C_1 &= \cosh\left(\frac{\kappa_1\pi}{b}\right) & c_1 &= \cos(p\pi) \\ S_1 &= \sinh\left(\frac{\kappa_1\pi}{b}\right) & s_1 &= \sin(p\pi) \end{aligned} \tag{19}$$

By expanding Equation (18) and by substituting Equation (15) into the result gives

$$1 - c_1C_1 + \frac{B}{D_x}\left[2\frac{B}{D_x}\frac{p^2b^2}{q^2a^2} + \frac{p^4b^4}{q^4a^4}\right]^{\frac{1}{2}} s_1S_1 = 0 \tag{20}$$

Second Auxiliary Problem

Solution of the second auxiliary problem of Equation (4) can be expressed as

$$W(x,y) = Y(y)\sin\left(\frac{p\pi x}{a}\right) \tag{21}$$

which satisfies the boundary condition

$$Y(y) = \frac{dY(y)}{dy} = 0 \text{ at y=0,b} \tag{22}$$

The solution of this problem can be obtained from the solution of the first auxiliary problem. The characteristic equation for the second auxiliary problem can be expressed as

$$\lambda^4 - 2\frac{B}{D_y}\left(\frac{p\pi}{a}\right)^2\lambda^2 - \frac{q^2\pi^2}{b^2}\left[2\frac{B}{D_y}\frac{p^2\pi^2}{a^2} + \frac{q^2\pi^2}{b^2}\right] = 0 \tag{23}$$

which has two imaginary and two real roots, namely

$$\lambda_{1,2} = \pm\frac{\pi}{ab}\kappa_2 \qquad \lambda_{3,4} = \pm i\frac{q\pi}{b} \tag{24}$$

where

$$\kappa_2 = \sqrt{\frac{2p^2b^2B}{D_y} + q^2a^2} \tag{25}$$

The general solution of this second auxiliary problem can be represented as

$$Y(y) = B_1\cosh(\kappa_2\pi\eta_2) + B_2\sinh(\kappa_2\pi\eta_2) + B_3\cos(q\pi\eta_2 a) + B_4\sin(q\pi\eta_2 a) \tag{26}$$

where B_j are constants of integration and $\eta_2 = \frac{y}{ab}$.

By substituting Equation (26) into Equation (23), one obtains the following characteristic determinant

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & \frac{\kappa_2}{a} & 0 & q \\ C_2 & S_2 & c_2 & s_2 \\ \frac{\kappa_2}{a}S_2 & \frac{\kappa_2}{a}C_2 & -qs_2 & qc_2 \end{vmatrix} = 0 \tag{27}$$

where

$$C_2 = \cosh\left(\frac{\kappa_2 \pi}{a}\right) \qquad c_2 = \cos(q\pi)$$
$$S_2 = \sinh\left(\frac{\kappa_2 \pi}{a}\right) \qquad s_2 = \sin(q\pi) \tag{28}$$

Expanding Equation (27) and substituting Equation (24) into the result gives

$$1 - c_2 C_2 + \frac{B}{D_y}\left[2\frac{B}{D_y}\frac{q^2 a^2}{p^2 b^2} + \frac{q^4 a^4}{p^4 b^4}\right]^{\frac{1}{2}} s_2 S_2 = 0 \tag{29}$$

FREQUENCIES AND MODE SHAPES

Equations (20) and (29) being transcendental in nature, have an infinite number of roots. The unknown quantities p and q are calculated from the solution of these equations. By substituting p and q into Equation (9), one obtains the eigen frequencies. The integer parts p and q represent the numbers of the eigen frequency. The mode shapes are determined as the product

$$W_{pq}(x, y) = X(x)Y(y) \tag{30}$$

where

$$X(x) = \cosh(\kappa_1 \pi \eta_1) - \cos(p\pi\eta_1 b) + \left[\frac{c_1 - C_1}{S_1 - \frac{\kappa_1}{bp} s_1}\right]\left[\sinh(\kappa_1 \pi \eta_1) - \frac{\kappa_1}{bp}\sin(p\pi\eta_1 b)\right] \tag{31}$$

and

$$Y(y) = \cosh(\kappa_2 \pi \eta_2) - \cos(q\pi\eta_2 a) + \left[\frac{c_2 - C_2}{S_2 - \frac{\kappa_2}{aq} s_2}\right]\left[\sinh(\kappa_2 \pi \eta_2) - \frac{\kappa_2}{aq}\sin(q\pi\eta_2 a)\right] \tag{32}$$

HOMOGENEOUS SOLUTION

The homogeneous solution of the problem can be obtained by a method of separation of variables. This technique is particularly useful for the direct solution of boundary value problems, where the boundary conditions have a simple form. The procedure comprises the derivation of a sequence of solutions of a separable form, in such a way that superposition yields a solution satisfying the boundary conditions. According to this method, the general solution of Equation (1) is set to be separated into functions of space and time,

$$
\begin{aligned}
w_{mn}(x,y,t) &= \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} X_m(x)Y_n(y)T_{mn}(t) \\
&= \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} X_m(x)Y_n(y)e^{-\bar{\gamma}\omega_{mn}t}\left[a_{mn}\cos\left(\sqrt{1-\bar{\gamma}^2}\,\omega_{mn}t\right)+b_{mn}\sin\left(\sqrt{1-\bar{\gamma}^2}\,\omega_{mn}t\right)\right]
\end{aligned}
\tag{33}
$$

in which $X_m(x)$ and $Y_n(y)$ are the spatial functions expressed by Equations (31) and (32), $T_{mn}(t)$ is the temporal function, ω_{mn} is the natural frequency of the system, $\bar{\gamma}$ is the damping ratio, a_{mn} and b_{mn} are constants, m and n are numbers of modes in x and y direction respectively that are equaled to the integer parts of p and q. The eigenvalues, denoted by α_{mn} are related to the undamped natural frequencies of vibration of the plate through

$$
{\alpha_{mn}}^4 = \rho h\left(\omega_{mn}\right)^2 \tag{34}
$$

DYNAMIC RESPONSE OF THE PLATE

Consider the case of the moving load with constant approaching speed v along the x direction. The load may be expressed as $P_0\cos\omega t$. At $t = t_0$, in which $t_0 = a/v$, the load leaves the plate. Thus, this problem may be treated in two parts. The first part involves a harmonically oscillating concentrated transverse load moving in x direction at a constant y_o position. The second part, in which the load is no longer on the plate, involves a free vibration response of the system. The two parts of the problem are related through the boundary conditions. The motion of the plate at $t = t_0$ due to the load at $y = y_0$ becomes the initial condition of the plate at the subsequent instantaneous loading change at $t = t_0$.

Using the above principles, the motion during an interval of time in which the load is no longer on the plate can be computed. Assuming the motion has achieved steady state prior to the load leaving the plate, the motion at $t = t_0$ may be easily computed. This motion at $t = t_0$ determines the initial condition for the second part of the problem. The response of the system can be easily computed by the following equation:

$$
w_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-f_4(t-t_0)}\left[w_{0mn}\cos[f_1(t-t_0)] + \frac{v_{0mn}+\bar{\gamma}\omega_{mn}w_{0mn}}{\sqrt{1-\bar{\gamma}^2}\,\omega_{mn}}\sin[\sqrt{1-\bar{\gamma}^2}\,\omega_{mn}(t-t_0)]\right] \tag{35}
$$

in which w_{omn} and v_{omn} in Equation (35) are the initial deflection and velocity at $t = t_0$.

Bending moments and the vertical shear forces in the plate can be computed in terms of the deflections obtained from Equation (35) from the following expressions:

$$
\begin{aligned}
M_x &= -\left[D_x\frac{\partial^2 w}{\partial x^2} + B\frac{\partial^2 w}{\partial y^2}\right] \quad ; \quad M_y = -\left[D_y\frac{\partial^2 w}{\partial y^2} + B\frac{\partial^2 w}{\partial x^2}\right] \\
Q_x &= -\frac{\partial}{\partial x}\left[D_x\frac{\partial^2 w}{\partial x^2} + H\frac{\partial^2 w}{\partial y^2}\right] \quad ; \quad Q_y = -\frac{\partial}{\partial y}\left[D_x\frac{\partial^2 w}{\partial y^2} + H\frac{\partial^2 w}{\partial x^2}\right]
\end{aligned}
\tag{36}
$$

where H=B+2G and G is the elastic shear modulus of the plate. In terms of elasticity moduli and Poison's ratios, the flexural rigidities and the effective torsional rigidity can be expressed as follows :

$$D_x = \frac{E_x\, h^3}{12\left(1-\nu_x\,\nu_y\right)} \quad ; \quad D_y = \frac{E_y\, h^3}{12\left(1-\nu_x\,\nu_y\right)} \quad ; \quad B = \sqrt{D_x D_y} \tag{37}$$

where E_x and E_y are the elasticity moduli in the x and y direction respectively, ν_x and ν_y are the Poison's ratios in the x and y direction respectively and h the thickness of the plate.

NUMERICAL EXAMPLE

Using the procedure of the last section, some results have been obtained for a rectangular clamped orthotropic plate subjected to a dynamic moving load. It is shown how the total dynamic deflections are affected by the load's frequency ω and by the damping ratio ($\overline{\gamma}$). Three cases have been calculated for which $\overline{\gamma}$ is equal to 0, 0.05 and 0.1, representing the damping factors for engineering structures. The transverse dynamic load is P_o = 1000 N, traveling with a constant speed v = 0.8 m/sec. along the x.

Table 1. Computed natural frequencies of the rectangular clamped orthotropic plate.

m	n	p	q	ω (rad/sec)
1	1	1.408	1.3678	15.1679
	2	1.295	2.4481	23.5972
	3	1.2352	3.4763	37.8947
	4	1.1379	4.4878	56.7479
	5	1.1113	5.4919	81.3999
2	1	2.4695	1.2321	36.6158
	2	2.4059	2.3659	44.6106
	3	2.3421	3.4281	57.9065
	4	2.2865	4.4543	76.4172
	5	2.2511	5.4692	100.4890
3	1	3.4846	1.1951	69.1955
	2	3.4488	2.3041	76.9268
	3	3.4062	3.3674	89.7048
	4	3.3572	4.4132	107.5950
	5	3.3292	5.4382	131.2740
4	1	4.4912	1.1612	112.5750
	2	4.4698	2.2311	119.9670
	3	4.4391	3.3221	132.7480
	4	4.4036	4.3699	150.2760
	5	4.3684	5.4042	173.0160

The following numerical results have been calculated for the case of a thin rectangular clamped orthotropic plate with dimensions and characteristics as follows: a=15m, b=20m, ρ=2400kg/m^3, h=0.12m, E_x=30x10^{9}N/m^2, E_y=20x10^{9}N/m^2, υ_x=0.2, υ_y=0.1, G=10^{10}N/m^2.

Based on the data shown above, the natural frequencies of the plate for the first 4 modes in x direction (m=1,2,…,4) and the first 5 modes in y direction (n=1,2,…,5) are calculated.

Table 1 shows the natural frequencies of the system for the first 4 modes in x direction (m=1,2,…,4) and the first 5 modes in y direction (n=1,2,…,5). It can be seen from the table that the natural frequency increases as the mode number increases.

Figure 2 shows the dynamic response spectra as a function of the load's frequency and damping ratio. It can be seen that the dynamic deflection will be maximum when the load's frequency approaches the value of the first natural frequency of the orthotropic plate.

Figure 3 gives an overview of the dynamic deflection shapes due to the transverse moving dynamic load.

Finally, Figure 4 shows the various responses of the clamped orthotropic plate to the moving transverse load. By comparing the case at near resonance condition and that away from resonance condition, one can recognize the significance of avoiding the resonance condition, since at resonance the various responses are apparently relatively very high.

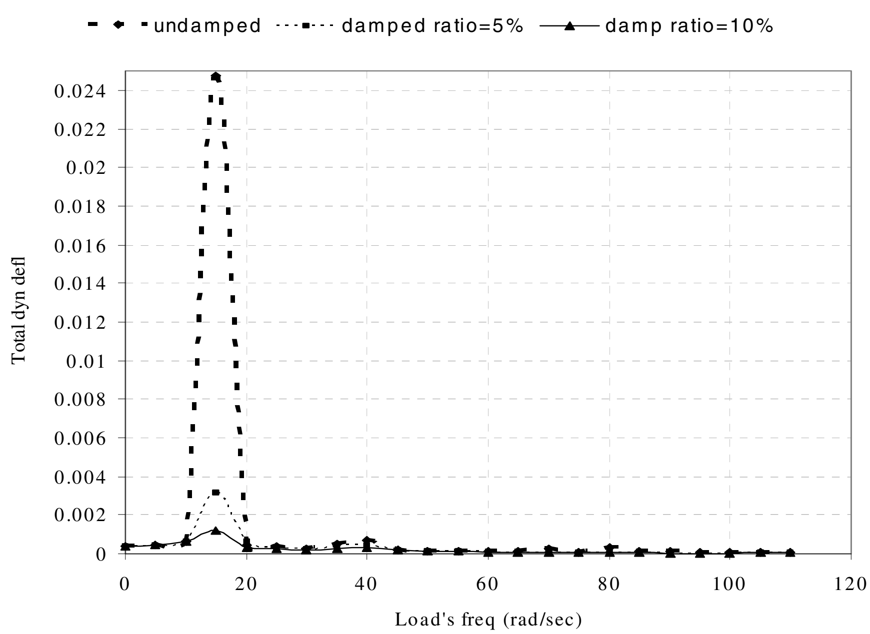


Figure 2. Maximum dynamic deflection response spectra for various values of load's frequency. Load's parameter: P_0=1000N, t_0=18.75sec, v=0.8 m/sec, y_0=5m.

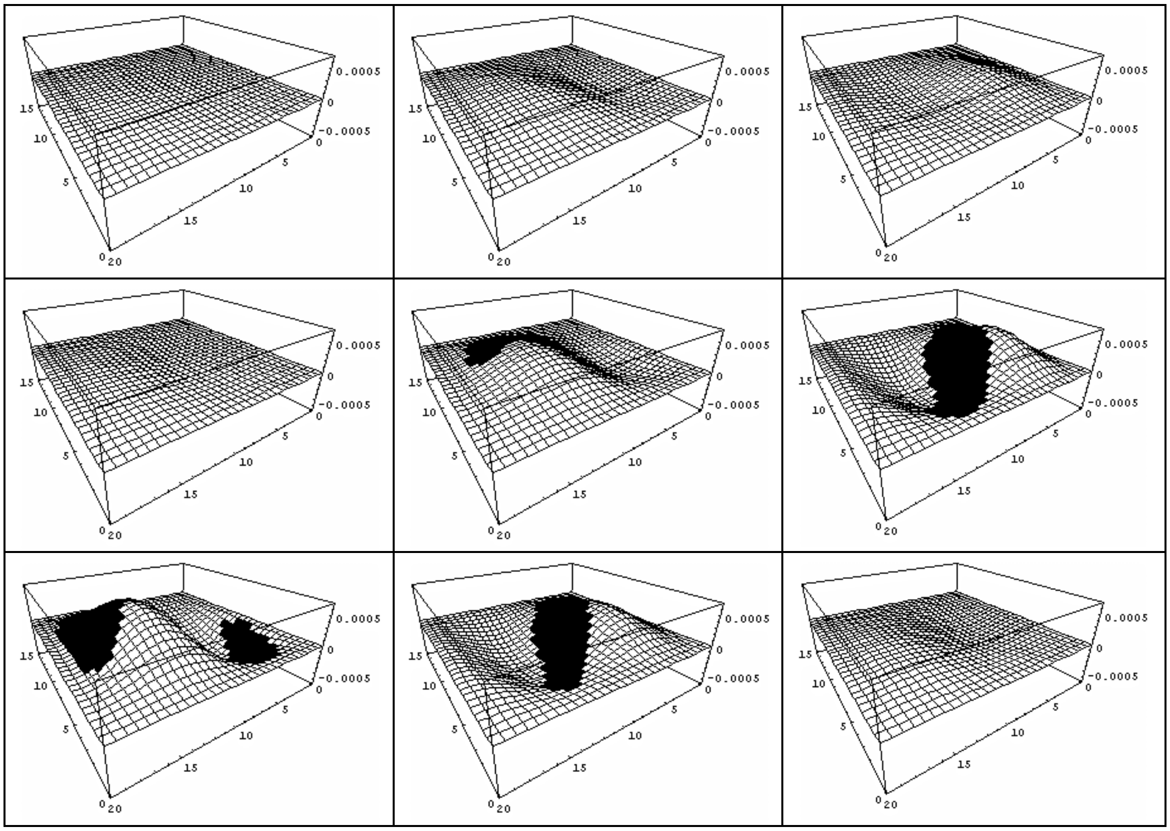
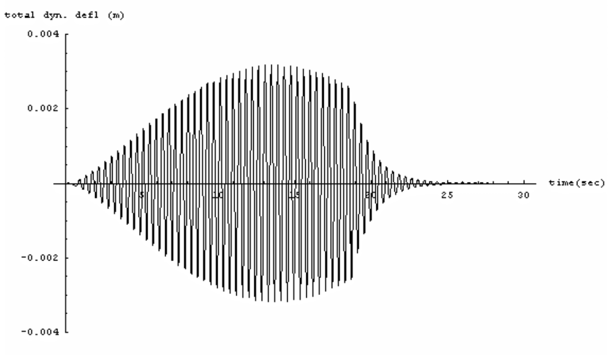
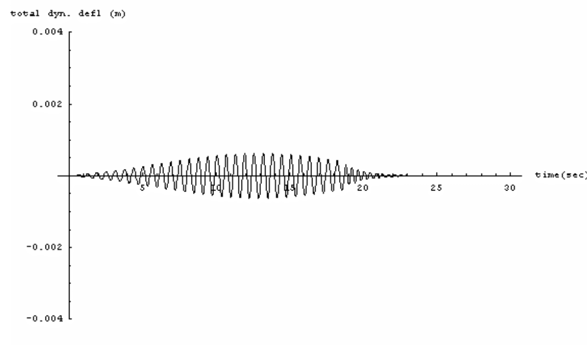


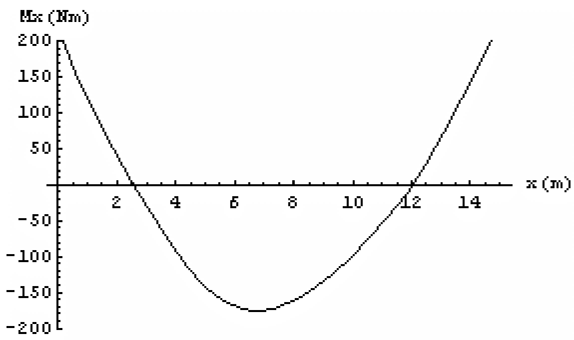
Figure 3. Examples of the total dynamic deflection mode shapes of an orthotropic rectangular plate clamped on all sides during the interval $t<t_0$ for m=1,2,…,4 and n=1,2,…,5. Load's parameter: P_0=1000N, ω=20 rad/sec, t_0=18.75 sec, $\overline{\gamma}$=5%, v=0.8 m/sec., y_0=5m.



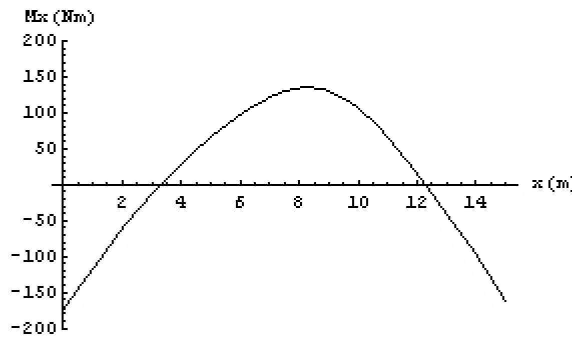
$\overline{\gamma}$=5%, ω=15 rad/sec



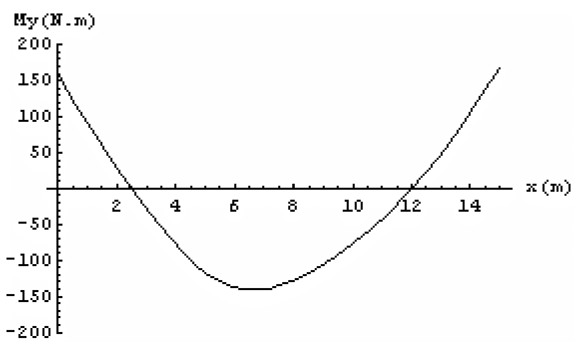
$\overline{\gamma}$=5%, ω=20 rad/sec



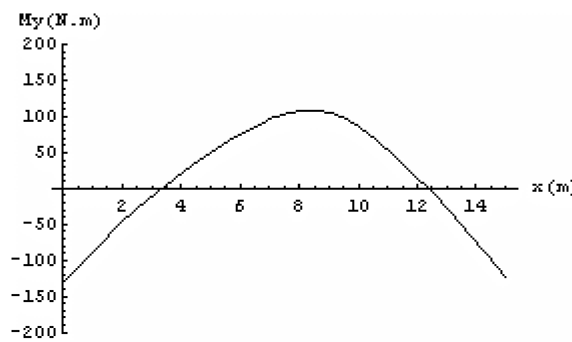
$\overline{\gamma}$=5%, ω=15 rad/sec



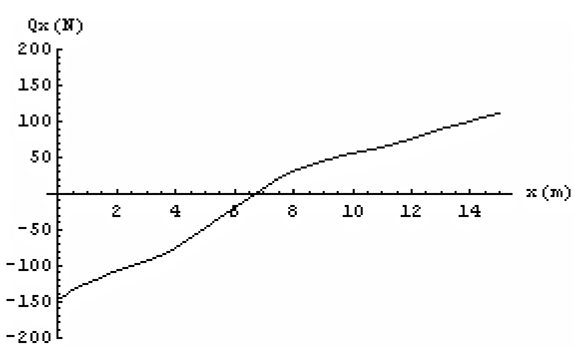
$\overline{\gamma}$=5%, ω=20 rad/sec



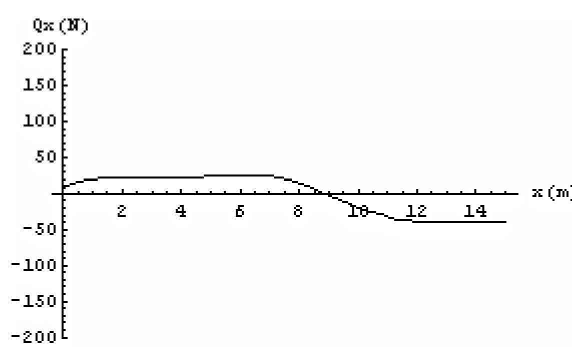
$\overline{\gamma}$=5%, ω=15 rad/sec



$\overline{\gamma}$=5%, ω=20 rad/sec



$\overline{\gamma}$=5%, ω=15 rad/sec



$\overline{\gamma}$=5%, ω=20 rad/sec

Figure 4. Various dynamic responses of the plate at near resonance condition (left) and away from resonance condition (right).

CONCLUSION

Based on the above study, the following conclusions can be drawn.

The use of two transcendental equations and the use of two Levy's type solutions to approximate natural frequencies and mode shapes of vibrating clamped rectangular orthotropic plates lead to very accurate results. This approximation is more convenient than the Bolotin's method, for example, because it provides a very simple way to derive the transcendental equations for the unknown wave numbers and leads to the determination of mode shapes for the entire region of the plate. The essential advantage of the presented method is the possibility of finding the frequency and the mode shape for any given pair of mode numbers.

The maximum dynamic deflection response spectra for various values of load's frequency may be used in the design of an orthotropic rectangular plate to determine response deflection, and to avoid the resonance condition.

REFERENCES

1. Alisjahbana, S.W., Dynamic Response and Stability Analysis of Orthotropic Plates on a Winkler Foundation, *Jurnal Teknik Sipil Untar*, No. 3, Th. VII, November 2001, pp. 379-401.
2. Alisjahbana S.W., X.L. Zhao, M. Alikhail, Dynamic Performance of Composite Beam Under Human Movement, *Jurnal Teknik Sipil Untar*, No. 2, Th. VI, July, 2000, pp. 141-153.
3. Alisjahbana, S.W., Dynamic Response of Elastically Supported Rectangular Plates to a General Surface Load, Jurnal *Teknik Sipil Untar* No. 1, Th VI, March, 2000, pp. 67-87.
4. Elishakoff, I.B., Vibration Analysis of Clamped Square Orthotropic Plate, AIAA Journal, Vol. 12, No. 7, July, 1974, pp. 921-924.
5. Leissa, A.W., *Vibration of Plates*, NASA SP-160, 1969.