Modul Panduan - Linear System Solution

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Modul Panduan - Linear System Solution

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1 Solving linear systems

Linear systems are fundamental concepts in mathematics and engineering, encompassing a set of linear equations that can be represented and solved using various methods. These systems are widely used to model real-world problems and relationships between variables. A linear system can be expressed in matrix form as $\mathbf{Ax} = \mathbf{b}$, where A is the coefficient matrix, x is the vector of unknowns, and b is the constant vector.

There are several approaches to solving linear systems, including substitution, elimination, and matrix methods. The graphical method is useful for visualizing two-dimensional systems, where the solution is represented by the intersection point of lines. For larger systems, more advanced techniques like Gaussian elimination and Gauss-Jordan elimination are employed to transform the augmented matrix into row echelon or reduced row echelon form.

The nature of solutions in linear systems can vary. A system may have a unique solution, infinitely many solutions, or no solution at all. The number of solutions is determined by the relationship between the equations and can be analyzed using concepts such as linear independence, rank, and determinants. Understanding these properties is crucial for interpreting the behavior of linear systems and their applications in various fields, from physics to economics.

1.1 Solving systems with substitution

Solving systems of equations using substitution is a fundamental algebraic technique that involves expressing one variable in terms of another and then substituting this expression into the remaining equation. This method is particularly effective when one of the equations can be easily rearranged to isolate a variable. The process typically involves four key steps:

1. selecting one equation and solving it for one of the variables
2. substituting this expression into the other equation
3. solving the resulting equation for the remaining variable
4. using this solution to find the value of the first variable

While substitution can be applied to systems with any number of equations and variables, it is most commonly used for systems of two equations with two unknowns. This approach not only provides a systematic way to find solutions but also helps develop algebraic manipulation skills and reinforces the concept of equivalent equations.

The following subsections describe how the system can be resolved by the substitution method:

$$2x + y = 7$$

$$x - 3y = -5$$

Step 1: Solve for one variable in terms of the other Solve for x in terms of y using the first equation:

$$x = \frac{7 - y}{2}$$

Step 2: Substitute into the second equation Replace x in the second equation:

$$\frac{7 - y}{2} - 3y = -5$$

Step 3: Solve for y Multiply everything by 2 to clear the fraction:

$$(7 - y) - 6y = -10$$

$$7 - 7y = -10$$

$$-7y = -17$$

$$y = \frac{17}{7}$$

Step 4: Substitute y back into $x = \frac{7-y}{2}$

$$x = \frac{7 - \frac{17}{7}}{2}$$

$$x = \frac{\frac{49}{7} - \frac{17}{7}}{2}$$

$$x = \frac{\frac{32}{7}}{2}$$

$$x = \frac{16}{7}$$

Solution:

$$x = \frac{16}{7}, \quad y = \frac{17}{7}$$

1.2 Solving systems with elimination

Solving systems of equations using elimination is an effective method for finding solutions to linear systems. Here are the key steps:

1. Rearrange both equations so that the x-terms are first, followed by the y-terms, then the equals sign and constant term.
2. Multiply one or both equations by constants that will allow either the x-terms or y-terms to cancel when the equations are added or subtracted.
3. Add or subtract the equations to eliminate one variable.
4. Solve for the remaining variable.
5. Plug the result from step 4 into one of the original equations and solve for the other variable.

This method works by strategically eliminating one variable, allowing you to solve for the other variable more easily. It is particularly useful for systems where substitution may be more complex. The elimination method can be extended to systems with more than two variables as well.

The system is solved using the following elimination method:

$$2x + 3y = 8$$
$$4x - y = -2$$

Step 1: Multiply to Make the Coefficients of y Equal Multiply the second equation by 3:

$$2x + 3y = 8$$
$$12x - 3y = -6$$

Step 2: Add the Equations to Eliminate y

$$(2x + 3y) + (12x - 3y) = 8 + (-6)$$

$$14x = 2$$

$$x = \frac{1}{7}$$

Step 3: Substitute x back into one of the original equations Using $4x - y = -2$:

$$4\left(\frac{1}{7}\right) - y = -2$$

$$\frac{4}{7} - y = -2$$

$$-y = -2 - \frac{4}{7}$$

$$-y = -\frac{14}{7} - \frac{4}{7}$$

$$-y = -\frac{18}{7}$$

$$y = \frac{18}{7}$$

Final Solution:

$$x = \frac{1}{7}, \quad y = \frac{18}{7}$$

1.3 Graphing method

The graphing method is a visual approach to solving systems of linear equations in two variables. To use this method, each equation is first rearranged into slope-intercept form ($y = mx + b$). Then, both equations are graphed on the same coordinate plane by plotting their y-intercepts and using their slopes to determine additional points. The solution to the system is represented by the point of intersection between the two lines. If the lines intersect at a single point, the system has one unique solution. If the lines are parallel and do not intersect, the system has no solution. If the lines completely overlap, the system has infinitely many solutions. While simple and intuitive for two-variable systems, the graphing method becomes impractical for systems with three or more variables, as these require three-dimensional or higher-dimensional graphs. For such cases, algebraic methods like substitution or elimination are preferred.

2 Matrix dimensions and entries

A matrix is a rectangular array of numbers arranged in rows and columns. The dimensions of a matrix are typically described as "rows $\times$ columns". For example, a 3x4 matrix has 3 rows and 4 columns.

Matrix entries are the individual numbers within the matrix. They are usually denoted by lowercase letters with subscripts indicating their position. For instance, a_{ij} represents the entry in the i^{th} row and j^{th} column.

Here is an example of a 3x3 matrix with its entries labeled:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The entries a_{11}, a_{12}, a_{13} form the first row, while a_{11}, a_{21}, a_{31} form the first column.

Understanding matrix dimensions and entries is crucial for performing matrix operations like addition, multiplication, and finding determinants. The dimensions determine which operations are possible between matrices, while the entries are used in the actual calculations.

2.1 Matrix addition

Matrix addition is a fundamental operation in linear algebra that involves combining two matrices of the same dimension by adding their corresponding elements. This operation is essential for manipulating and analyzing data represented in matrix form. To perform matrix addition, the matrices must have an equal number of rows and columns, ensuring that they are compatible for operation.

The matrix addition process is straightforward: each element in the resulting matrix is obtained by adding the corresponding elements from the two input matrices. For example, if we have two matrices A and B, then the sum matrix C is calculated as

$$C[i, j] = A[i, j] + B[i, j]$$

where i represents the row and j represents the column. The resulting matrix has the same dimensions as the input matrices.

One important property of matrix addition is that it is commutative, meaning the order of addition does not affect the result. In other words, $A + B = B + A$. This property allows for flexibility in calculations and simplifies many mathematical proofs involving matrices. In addition, matrix addition is associative, which means that $(A + B) + C = A + (B + C)$. These properties make matrix addition a well-behaved operation in linear algebra. Consider the matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

The sum $C = A + B$ is given by:

$$C = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

2.2 Matrix subtraction

Subtracting elements from matrices of the same size is a fundamental operation in linear algebra, and it requires subtracting the corresponding elements. Performing operations on data in matrix form is essential for analysis and manipulation. For matrix subtraction to be valid, it is essential that the involved matrices possess the same number of rows and columns, and this condition is a necessary prerequisite for the operation to be mathematically well-defined.

The matrix subtraction process is uncomplicated and involves subtracting each element in the corresponding position of the second matrix from the first matrix. The element-wise subtraction produces a new matrix with the same dimensions as the subtracted matrices. The operation is represented by the minus sign (-) positioned between the two matrices, which represents the subtraction of one matrix from the other.

The matrix subtraction does not follow the commutative property. This demonstrates that A minus B is not equal to B minus A, where A and B are matrices. The order in which matrices are subtracted can yield distinct results depending on which matrix is subtracted from the other. Consider the matrices:

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix}$$

The result of $C = A - B$ is:

$$C = \begin{bmatrix} 4-1 & 7-3 \\ 2-6 & 5-2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

2.3 Scalar multiplication

Linear algebra relies on scalar multiplication operations, which are essential when working with vectors and matrices. This operation entails multiplying a vector or matrix by a single numerical quantity, specifically a scalar value, which can be an integer or a real number. Scalar multiplication is important to grasp and execute multiple mathematical transformations in complex multidimensional environments.

Multiplying a scalar by a vector results in each component of the vector being scaled by that scalar value. The scaling process produces a new vector that retains the same direction as the original vector but possesses a distinct magnitude. Multiplying a vector with coordinates (2, 3, 4) by a scalar of 2 produces a new vector with coordinates (4, 6, 8). This example illustrates how multiplying a vector by a scalar function impacts its magnitude while maintaining its direction in space.

In matrix operations, scalar multiplication extends this principle to two-dimensional arrays of numbers. When a scalar is multiplied by a matrix, it is applied to every element in the matrix, effectively scaling the entire structure. The uniform scaling of matrix elements is particularly useful for various mathematical transformations and computations involving linear systems.

Scalar multiplication has several important properties that make it a versatile mathematical analysis tool. First, it is commutative, meaning that the order of multiplication between a scalar and a vector or matrix does not affect the result. Second, it is associative, allowing for the grouping of multiple scalar multiplications without changing the outcome. Finally, scalar multiplication is distributive over vector addition, which means that it can be applied to the sum of vectors or matrices by distributing the scalar to each term individually.

These properties make scalar multiplication an indispensable tool for computer-graphics applications in various fields. For example, scalar multiplication can be used to scale objects, adjust brightness levels, and modify color intensities. In physics simulations, it helps model the effects of forces and other physical quantities on objects in motion. Data analysts employ scalar multiplication to normalize data sets, adjust weights in machine learning algorithms, or perform dimensionality reduction techniques. Consider the matrix:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & -5 \end{bmatrix}$$

Let $k = 3$ be a scalar. The scalar multiplication kA is computed as:

$$3A = 3 \times \begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 3 \times 1 & 3 \times (-2) & 3 \times 3 \\ 3 \times 4 & 3 \times 0 & 3 \times (-5) \end{bmatrix} = \begin{bmatrix} 3 & -6 & 9 \\ 12 & 0 & -15 \end{bmatrix}$$

2.4 Matrix multiplication

Matrix multiplication is a fundamental operation that underpins linear algebra, and it has widespread applications across multiple disciplines, including mathematics, physics, computer science, and engineering. The powerful mathematical tool requires the multiplication of two matrices, resulting in a new matrix, which is achieved through a strict and precise rule-based combination of individual elements. This process is crucial not only for understanding theoretical concepts but also for resolving practical problems in various fields.

For two matrices to be multiplied together, a fundamental requirement must be fulfilled: the number of columns in the first matrix must exactly match the number of rows in the second matrix. This requirement ensures that the multiplication operation is properly defined and executed. The resulting matrix (referred to as the product matrix) has dimensions that match the number of rows in the first matrix and the number of columns in the second matrix. Understanding the structure and properties of the resulting matrix requires knowledge of its dimensional relationships.

The calculation of each element in the product matrix follows a specific procedure. The second method takes the dot product of a row from the first matrix and a column from the second matrix. This process requires multiplying the corresponding elements and then summing the results. The dot product calculation is repeated for each combination of rows from the first matrix and columns from the second matrix, filling in the elements of the product matrix one by one.

An important characteristic of matrix multiplication is that it does not involve a commutative process. Thus, changing the order of multiplication yields different results. In other words, for matrices A and B, A × B is not necessarily equal to B × A. This property distinguishes matrix multiplication from scalar multiplication and addition, which are commutative operations. The non-commutative nature of matrix multiplication has significant implications for various applications and theoretical considerations.

Matrix multiplication is a vital component in computer graphics and computer vision applications, facilitating the conversion of coordinates and linear transformations. This feature allows the rotation, scaling, and translation of objects in two- and three-dimensional settings. This capability is crucial for producing realistic computer-generated images and video game development. This capability also involves the implementation of computer vision algorithms.

Consider the matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

The product $C = A \times B$ is given by:

$$C = \begin{bmatrix} (1 \times 5 + 2 \times 7) & (1 \times 6 + 2 \times 8) \\ (3 \times 5 + 4 \times 7) & (3 \times 6 + 4 \times 8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

2.5 Zero and identity matrices

A zero matrix, denoted as 0, is a matrix in which all elements are zero, regardless of its dimensions. This unique property makes it particularly useful in mathematical modeling and computation. For instance, in graph theory, a zero matrix can represent a graph with no edges, whereas, in economics, it might indicate a scenario with no interactions between variables.

The zero matrix serves as additive identity in matrix operations, which is analogous to the zero number in scalar arithmetic. This means that adding a zero matrix to any other matrix results in the original matrix, which is a crucial property in matrix algebra. Furthermore, the zero matrix is the only matrix that always yields another zero matrix when multiplied by any other matrix. This property is often used in the proofs and theoretical derivations of advanced linear algebra.

In contrast, an identity matrix, typically represented by I, is a square matrix with ones on the main diagonal and zeros elsewhere. The identity matrix is the cornerstone of linear transformations and matrix theory. The structure of this method makes it invaluable for preserving vector magnitudes and directions when used in matrix multiplication. In computer graphics and 3D modeling, an identity matrix often serves as the starting point for transformation matrices.

The identity matrix acts as the multiplicative identity for matrices, similar to how the number one functions in scalar multiplication. This property is essential in matrix inversion, where the product of a matrix and its inverse yields the identity matrix. The concept of the identity matrix extends to more abstract mathematical structures, such as group theory, where it plays a role analogous to the number one in arithmetic.

Both zero and identity matrices play crucial roles in solving systems of linear equations, matrix transformations, and vector spaces. In linear equation systems, the zero matrix can represent a homogeneous system, whereas the identity matrix is often used in Gaussian elimination to solve for unknown variables. In vector spaces, these matrices help define important concepts like null spaces and eigenvalues.

Their properties are essential for simplifying complex matrix operations, and they are frequently used in various fields, such as mathematics, physics, and engineering. For example, in quantum mechanics, an identity matrix represents the quantum state of a system that remains unchanged, whereas zero matrices can describe the absence of certain quantum interactions. In control theory, identity matrices are used to model ideal systems without disturbances, and zero matrices can represent the absence of control inputs. Here is the example of the 3×3 identity matrix is given by:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 Augmented matrix

A matrix representation, commonly known as an augmented matrix, is a crucial component in linear algebra that unifies the coefficients of a system of linear equations with the constants present in the equations to form a comprehensive and singular representation. This format enables a more efficient approach to solving systems of equations, as it consolidates all pertinent information into a single, unified framework. The augmented matrix is usually represented by a vertical line that separates the coefficient matrix from the constant terms, allowing for easy differentiation between the two components.

Utilizing augmented matrices provides several benefits in the resolution of mathematical problems. This approach simplifies the application of row operations by enabling all necessary transformations to be carried out on a single matrix, rather than separate equations. The efficiency of this approach is especially advantageous in complex systems with numerous variables. Augmented matrices allow for the efficient application of Gaussian elimination and other matrix reduction methods, thus providing a structured approach to obtaining solutions. Augmented matrices offer a clear visual representation of a system, enabling the identification of inconsistent or dependent equations, which is essential for determining the nature and existence of solutions in linear systems.

Key points about augmented matrices:

1. Structure: An augmented matrix is formed by combining coefficient matrix A and constant vector

b into a single matrix, denoted as $[A|b]$.

2. Vertical line: The coefficient is distinguished from the constant by a vertical line.
3. Dimensions: An augmented matrix for a system with m equations and n variables comprises m rows and $n+1$ columns.
4. Row operations: Elementary row manipulations can be applied to the augmented matrix to solve the system.
5. Reduced row echelon form: The transformation of the augmented matrix to a reduced row echelon form facilitates the determination of the solution set.
6. Solution interpretation: The reduced augmented matrix's far-right column indicates the solution or conditions for obtaining solutions.
7. Consistency: The augmented matrix can be used to determine whether a system of equations has consistent or inconsistent solutions.
8. Gaussian elimination: This method employs an augmented matrix to systematically remove variables and find solutions to the given system.
9. Matrix equation: The augmented matrix $[A|b]$ represents the matrix equation $Ax = b$.
10. Parametric solutions: The augmented matrix of systems with infinite solutions can be expressed in parametric form.

Solving linear equation systems can be achieved effectively by combining a coefficient matrix with a constant term matrix, resulting in a unified and efficient matrix structure. This unified representation allows us to apply row operations more efficiently and implement elimination methods. Mathematicians and students can simplify complex analysis by consolidating all relevant information into a single matrix, enabling them to more effectively visualize the interdependencies between variables and constants and gain a clearer comprehension of the issue.

Using augmented matrices streamlines the computational process while improving the capacity to detect patterns and connections within a system of equations. This approach is especially advantageous for systems with many variables and equations. Augmented matrices provide a basis for further linear algebra techniques, including Gaussian elimination and matrix transformations. Individuals can enhance their problem-solving abilities in various mathematical and scientific fields by gaining a deeper understanding of linear systems through proficiency in augmented matrices. The augmented matrix typically takes the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & | & b_2 \\ \vdots & \vdots & & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & | & b_m \end{bmatrix}$$

3.1 Pivot entries and pivot columns

Pivot elements are also referred to as pivot entries and are important in linear algebra and matrix operations. During Gaussian elimination or matrix factorization, entries are chosen to convert a matrix into a row echelon form or to solve systems of linear equations. A pivot element is generally a nonzero value within a matrix that is employed to nullify other elements in the column below it. The column holding the pivot entry is referred to as the pivot column.

In Gaussian elimination, the process involves selecting pivot entries one by one from left to right and then using them to generate zeros in the row below each pivot within the respective column. The process is repeated until the matrix forms a row or reduced row echelon. The selection of pivot entries has a substantial effect on the numerical stability and efficiency of matrix computations, making it a crucial factor to consider in linear algebra algorithms and numerical methods.

Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

Pivot Entries: The first nonzero entry in each row:

- Row 1: 1 (first column)
- Row 2: 5 (third column)
- Row 3: 7 (fourth column)

Pivot Columns: The columns containing these pivot entries:

- Column 1 (because of pivot entry 1)
- Column 3 (because of pivot entry 5)
- Column 4 (because of pivot entry 7)

Thus, the pivot columns are: *Column 1, Column 3, and Column 4*.

3.2 Echelon forms

The echelon form is a fundamental concept in linear algebra and plays a crucial role in the study of matrices and systems of linear equations. These simplified matrix representations are powerful tools for efficiently analyzing and solving complex linear systems.

There are two primary types of echelon forms: row and reduced row forms. Each of these forms has specific characteristics and applications in linear algebra.

The row echelon form is characterized by the following key properties:

1. The leading coefficient (the first non-zero element) of each row is strictly to the right of the leading coefficient of the row above it.
2. Any row consisting of zeros is positioned at the bottom of the matrix.
3. The elements below each leading coefficient are zero.

The reduced row echelon form, also known as the row canonical form, is a more refined version of the row echelon form. It possesses all the properties of the row echelon form and additionally:

1. Each leading coefficient is equal to 1.
2. Each leading coefficient is the only nonzero entry in its column.

The process of transforming a matrix into either a row echelon form or reduced row echelon form involves a series of elementary row operations. These operations include the following steps:

1. Scaling: A row is multiplied by a nonzero scalar.
2. Addition: This step adds a multiple of one row to another.
3. Swapping: The positions of two rows.

These operations are performed systematically to simplify the matrix structure while preserving the solution set of the corresponding system of linear equations.

The echelon forms have numerous applications in linear algebra and related fields:

1. Determining matrix rank: The number of nonzero rows in the echelon form of a matrix directly corresponds to its rank, which is a crucial property in many linear algebra applications.
2. Solving systems of linear equations: Echelon forms simplify the process of solving linear systems by creating a triangular or diagonal structure that allows for straightforward back-substitution or forward-substitution techniques.
3. Computing matrix inverses: A matrix's reduced row echelon form is especially helpful in determining its inverse, if one exists, via a method that includes appending an identity matrix to the original matrix.
4. Analyzing linear independence: Echelon forms help determine whether a set of vectors is linearly independent or dependent, which is essential when studying vector spaces and subspaces.
5. Calculating determinants: Using the echelon form simplifies the process of determining matrix determinants because the determinant of a triangular matrix is the product of its diagonal elements.
6. Identifying basis vectors: In the study of vector spaces, echelon forms assist in finding basis vectors for the column, row, and null spaces of a matrix.

In computational linear algebra, algorithms have been developed to efficiently calculate echelon forms, rendering them useful for numerical analysis and scientific computation. Many mathematical software tools and programming libraries incorporate these algorithms, enabling researchers and engineers to tackle large-scale linear algebra problems efficiently.

3.3 Gauss-Jordan elimination, Gaussian elimination

Gaussian elimination, also known as row reduction, is a systematic approach to transform a matrix into a row echelon form through a series of elementary row operations. This process involves methodically eliminating variables to create an upper-triangular matrix. The procedure begins with the leftmost nonzero column and works its way to the right, using pivots to eliminate coefficients below the diagonal. The resulting row echelon form facilitates analysis and solution.

Gauss-Jordan elimination builds upon Gaussian elimination, achieving a more refined row echelon form through extra row operations. This matrix features ones on the primary diagonal and zeros elsewhere, thereby establishing a simpler structure that directly represents the solution to the system. This approach not only removes coefficients below and above the diagonal, thereby yielding a diagonal matrix with elements along its main diagonal.

Gauss-Jordan elimination has the advantage of yielding the solution directly compared to Gaussian elimination, which necessitates back-substitution as an additional step. This direct approach is especially beneficial when handling multiple right-hand sides or when the inverse of the coefficient matrix is required.

Gaussian elimination typically provides greater computational efficiency, particularly in large-scale systems. Achieving the row echelon form requires fewer arithmetic steps. It is a more suitable choice when computational resources are limited or when working with large datasets. The choice between these two methods usually hinges on the problem's specific needs and the computational resources that are available.

These elimination techniques have far-reaching applications beyond simply solving linear equations. They are fundamental in various mathematical and scientific computations, including the following:

1. Matrix inversion: Gauss-Jordan elimination can be used to find the inverse of a matrix by augmenting the original matrix with an identity matrix and reducing it to reduced row echelon form.

2. Determinant calculation: The row operations performed during Gaussian elimination can be used to simplify determinant calculations.

3. Rank determination: The number of nonzero rows in the row echelon form obtained through Gaussian elimination determines the rank of the matrix.

4. Basis of vector spaces: These methods can help identify the basis of vector spaces and subspaces.

5. Least-squares fitting: In data analysis and curve fitting, these techniques are used to solve normal equations using the least-squares method.

As computational capabilities have advanced, variations and optimizations have been developed to handle increasingly large and complex systems. Parallel computing techniques have been applied to distribute the computational load of these algorithms across multiple processors, enabling the solution of massive systems of equations that were previously intractable.

The following is row echelon form steps. Consider the matrix:

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$

Step 1: Make the first pivot 1 (Divide the first row by 2):

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$

Step 2: Eliminate the first column entries below the pivot Perform:

$$R_2 \leftarrow R_2 - R_1, \quad R_3 \leftarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix}$$

Step 3: Make the second pivot 1 and eliminate below it Perform:

$$R_3 \leftarrow R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

This is the **row echelon form** of the matrix.

The following is the example of the linear system is solved by using Gauss-Jordan elimination:

$$\begin{bmatrix} 2 & 1 & -1 & | & 8 \\ -3 & -1 & 2 & | & -11 \\ -2 & 1 & 2 & | & -3 \end{bmatrix}$$

Step 1: Make the first pivot 1 (Divide R_1 by 2):

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & | & 4 \\ -3 & -1 & 2 & | & -11 \\ -2 & 1 & 2 & | & -3 \end{bmatrix}$$

Step 2: Eliminate below the first pivot Perform:

$$R_2 \leftarrow R_2 + 3R_1, \quad R_3 \leftarrow R_3 + 2R_1$$

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & |4 \\ 0 & \frac{1}{2} & \frac{1}{2} & |1 \\ 0 & 2 & 1 & |5 \end{bmatrix}$$

Step 3: Make the second pivot 1 (Multiply R_2 by 2):

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & |4 \\ 0 & 1 & 1 & |2 \\ 0 & 2 & 1 & |5 \end{bmatrix}$$

Step 4: Eliminate above and below the second pivot Perform:

$$R_3 \leftarrow R_3 - 2R_2, \quad R_1 \leftarrow R_1 - \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 0 & -1 & |3 \\ 0 & 1 & 1 & |2 \\ 0 & 0 & -1 & |1 \end{bmatrix}$$

Step 5: Make the third pivot 1 (Multiply R_3 by -1):

$$\begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

Step 6: Eliminate above the third pivot Perform:

$$R_1 \leftarrow R_1 + R_3, \quad R_2 \leftarrow R_2 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

Thus, the solution to the system is:

$$x = 2, \quad y = 3, \quad z = -1$$

4 Number of solutions to the system

The number of solutions to a system of equations can vary significantly depending on the nature of the equations and their relationships. For example, linear systems can have one unique solution, an infinite number of solutions, or no solution at all. These outcomes determine the geometric relationships among the equations in the system. A system with a unique solution occurs when the equations intersect at a single point in the solution space. This scenario is often the most straightforward and desirable in many applications because it provides a definitive answer to the given problem.

Infinite solutions arise when the equations represent coincident lines or planes, indicating that any point on that line or plane satisfies all equations simultaneously. This proves that the system is underdetermined, meaning that there are more unknowns than independent equations. While this may seem problematic, it can be advantageous in certain contexts, such as optimization problems where multiple optimal solutions exist.

Conversely, a system may have no solution if the equations are inconsistent, representing parallel lines or planes that never intersect. This scenario often indicates a contradiction within the system or an impossibility in the modeled real-life situation. Recognizing and interpreting such cases is crucial for avoiding erroneous conclusions or impossible designs in practical applications.

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